National Curriculum

Ma

Teaching mental calculation strategies to level 5

QCA

department for
education and skills
creating opportunity, releasing potential, achieving excellence

Key Stage 3
National Strategy

Teaching mental calculation strategies to level 5

Acknowledgements

The companion to this booklet is *Teaching mental calculation strategies: guidance for teachers at key stages 1 and 2* (© Qualifications and Curriculum Authority 1999; reference QCA/99/380).

The puzzles on page 30 are the copyright of Anita Straker and have appeared previously in the *Times Educational Supplement*. We are grateful for her permission to use them.

Pictures supplied under Education Photos terms and conditions. John Walmsley asserts his right to be identified as the author of these photos under the Copyright, Design and Patents Act 1988.

Disclaimer

The Department for Education and Skills wishes to make clear that the Department and its agents accept no responsibility for the actual content of any materials suggested as information sources in this document, whether these are in the form of printed publications or on a website.

In these materials icons, logos, software products and websites are used for contextual and practical reasons. Their use should not be interpreted as an endorsement of particular companies or their products.

The websites referred to in these materials existed at the time of going to print. Teachers should check all website references carefully to see if they have changed and substitute other references where appropriate.

First published in 2004

Printed in Great Britain

The Qualifications and Curriculum Authority is an exempt charity under Schedule 2 of the Charities Act 1993.

Qualifications and Curriculum Authority
83 Piccadilly
London
W1J 8QA
www.qca.org.uk

Contents

1 Introduction

The ability to calculate in your head is an important part of mathematics. It is also an important part of coping with society's demands and managing everyday events. The national curriculum and the *Framework for teaching mathematics: years 7, 8 and 9* make clear that pupils should learn number facts by heart and be taught a range of mental strategies for finding quickly facts that they cannot recall. Alongside this, pupils need to learn efficient methods for calculating with pencil and paper and with a calculator.

The purpose of this booklet

This booklet for key stage 3 teachers is adapted from a similar booklet for teachers in key stages 1 and 2. It is concerned mainly with pupils who achieve level 3 or a weak level 4 in the key stage 2 tests. The aim for them is to achieve level 5 by the end of key stage 3. In this booklet, these pupils are referred to as the target pupils.

When they start in key stage 3, the target pupils tend to use less efficient methods to calculate and their understanding of those methods may be incomplete. The purpose of this booklet is to offer guidance on teaching effective and efficient strategies for mental calculation. It is designed to assist your planning by:

- listing the number facts that pupils are expected to recall rapidly
- setting out expectations of the types of calculations that pupils should be able to do
- listing those strategies that might be taught to pupils to help them to calculate accurately and efficiently
- suggesting a range of suitable classroom activities to help pupils to develop and understand calculation methods.

Part 2

Part 2 describes the progression in the calculation facts and calculations that pupils are expected to know and be able to perform mentally. The expectations, which cover levels 3, 4 and 5 of the national curriculum, are adapted from expectations for years 4, 5 and 6 in the primary mathematics framework, and extended.

Part 3

Part 3 discusses choosing the most appropriate mental strategies for a range of calculations of varying degrees of difficulty. Discussing with pupils the relative merits of different strategies helps them to see why some strategies are more appropriate and efficient than others. Talking through a strategy will aid pupils' learning and sharpen their ability to perform calculations mentally. Key questions to ask include: 'How did you work that out?', 'Who did it another way?', 'Which is the easiest way?'.

You will need to stress the idea of the efficiency of a method – the ease and speed with which it leads to a correct answer. Different strategies may be used to perform a particular calculation, but some may be more efficient than others. Give pupils plenty of opportunities to discuss the relative merits of these strategies when they are applied to different computations.

Many of the target pupils are unable to hold in their head all the information they need when they are calculating mentally. For them, informal recording should be encouraged. Ask pupils to explain their mental calculation strategies and to record the results of a mental calculation in a number sentence, using the correct signs and symbols. This helps them to become accustomed to both spoken and written forms of a calculation.

Parts 4 and 5

Parts 4 and 5 specify the key mental strategies to teach pupils. They include activities to support the teaching of these strategies. Part 4 deals with addition and subtraction and Part 5 with multiplication and division. Each subsection begins with examples of typical problems.

Part 6

Part 6 discusses the use of a basic calculator. Calculator skills need to be taught. Pupils don't acquire these skills merely by having opportunities to use calculators.

Calculators don't replace the need for mental skills. They are useful for calculations that are too difficult to do mentally. Pupils also need to be competent in written calculation methods to use when calculators are not available.

Calculators can be very effective teaching tools, for example, to show number patterns such as multiplying or dividing by 10, and in reinforcing concepts in place value, such as that 367 is 300 + 60 + 7. There are also constructive ways to use calculators to explore and teach aspects of the structure of the number system.

Part 7

Part 7 deals with estimating and checking. It draws together ideas from all the previous sections to show how pupils can consolidate their learning and understanding of the mathematical principles set out in this booklet.

Glossary

A glossary of terms used in the main pages of this booklet is also included.

2 Mental calculations: progression

All key stage 3 teachers need to review, consolidate and build on pupils' mental calculation skills throughout years 7, 8 and 9. To help your planning, the mental strategies and skills that relate to whole-number calculations up to level 5 of the national curriculum are set out for you below. They are roughly graded in difficulty to show progression.

Addition and subtraction of whole numbers		
Rapid recall **Pupils should be able to recall rapidly:**	**Mental strategies** **Pupils should understand and be able to apply these key strategies:**	**Mental calculations** **Working mentally, pupils should be able to:**
• all pairs of numbers with a total of 10, eg 3 + 7 • all pairs of multiples of 10 with a total of 100, eg 30 + 70 all pairs of multiples of 5 with a total of 100, eg 35 + 65 • all pairs of multiples of 100 with a total of 1000, eg 300 + 700 • addition and subtraction facts for all numbers to 20, eg 9 + 8, 17 – 9	• count on or back in ones, tens or hundreds • find the sum of two numbers from the set 5, 6, 7, 8, 9 by partitioning each into '5 and a bit', adding the two fives, then adding the remaining parts • add or subtract 9 or 11 by adding or subtracting 10 and adjusting by 1 • add a single-digit number by bridging through a multiple of 10, eg 47 + 8 = 47 + 3 + 5 = 50 + 5 • reorder numbers in an addition, eg put the larger number first • add three small numbers by putting the largest number first and/or finding a pair totalling 10 (or 9 or 11) • find a small difference by counting up from the smaller to the larger number, eg 102 – 97 • add or subtract the nearest multiple of 10 then adjust, eg 56 + 29 = 56 + 30 – 1, 86 – 38 = 86 – 40 + 2 • partition two-digit additions into tens and units, add tens and units separately, then recombine • use knowledge of the inverse relationship between addition and subtraction • use related calculations, eg use 14 + 15 = 29 to work out 140 + 150 = 290	• add or subtract a single digit to or from a single digit, without crossing 10, eg 4 + 5, 8 – 3 • add or subtract a single digit to or from 10 • add or subtract a single digit to or from a two-digit number, without crossing the tens boundary eg 23 + 5, 57 – 3 • add or subtract any single digit to or from a multiple of 10, eg 60 + 5, 80 – 7 • add or subtract a single digit to or from a two-digit number, crossing the tens boundary, eg 28 + 5, 52 – 7 • find what must be added to any two-digit multiple of 10 to make 100, eg 70 + ? = 100 • add or subtract a multiple of 10 to or from any two-digit number, without crossing 100, eg 47 + 30, 82 – 50 • add or subtract a pair of two-digit numbers, without crossing the tens or hundreds boundaries, eg 34 + 65, 87 – 23 • add or subtract a pair of two-digit numbers, crossing the tens but not the hundreds boundaries, eg 38 + 45, 82 – 36 • add or subtract a pair of two-digit numbers, crossing the hundreds but not the tens boundaries, eg 38 + 81, 109 – 43 • add or subtract any pair of two-digit numbers

Multiplication and division of whole numbers		
Rapid recall **Pupils should be able to recall rapidly:**	**Mental strategies** **Pupils should understand and be able to apply these key strategies:**	**Mental calculations** **Working mentally, pupils should be able to:**
• doubles of all numbers to 10, eg 8 + 8, double 6 • multiplication facts to 10 × 10 • squares of all integers 1 to 10 • division facts corresponding to tables up to 10 × 10	• identify near doubles, eg 12 + 13 = double 12 plus 1 • multiply a number by 10 or 100 by shifting its digits one or two places to the left • divide a number by 10 or 100 by shifting its digits one or two places to the right • double a two-digit number by partitioning it into tens and units, doubling the tens, doubling the units, and recombining • use partitioning to multiply, eg 23 × 4 = (20 + 3) × 4 = (20 × 4) + (3 × 4) = 80 + 12 = 92 • use partitioning to divide, eg 92 ÷ 4 = (90 + 2) ÷ 4 = (80 + 12) ÷ 4 = 20 + 3 = 23 • use factors to multiply or divide, eg to multiply by 12, multiply by 3, then multiply by 4 to divide by 15, divide by 3, then divide by 5 • use knowledge of the inverse relationship between halving and doubling • use knowledge of the inverse relationship between multiplication and division • use related calculations, eg given 6 × 7 = 42, work out 60 × 70 = 4200 given 13 × 18 = 234, work out 13 × 19 = 234 + 13 = 247 • multiply or divide a number by 1000 by shifting its digits three places to the left or right • transform a multiplication into an equivalent calculation, eg to multiply by 25, multiply by 100, then divide by 4 to multiply by 50, multiply by 100, then halve	• double any number to 20, eg double 18 • halve any even number to 40 • double any multiple of 5 up to 50, eg double 35 • halve any multiple of 10 up to 100, eg halve 50 • multiply any one- or two-digit number by 10, 100 or 1000, eg 7 × 1000, 26 × 100 • divide any multiple of 10 by 10, eg 120 ÷ 10 • divide a multiple of 100 by 10 or 100, eg 400 ÷ 10, 3600 ÷ 100 • multiply a two-digit number by a single-digit number, first without and then with crossing the hundreds boundary, eg 32 × 3, 52 × 4 • divide a two-digit number by a single-digit number, no remainder eg 68 ÷ 4 • find the remainder when a two-digit number is divided by a single-digit number, eg 27 ÷ 4 = 6 R 3 • double any whole number from 1 to 50 • halve any even number to 100 • square any multiple of 10 to 100 • divide any whole number by 10, 100, giving the quotient as a decimal, eg 47 ÷ 10 = 4.7, 1763 ÷ 100 = 17.63 • divide any multiple of 1000 by 10, 100 or 1000, eg 16 000 ÷ 1000, 4000 ÷ 100

3 Teaching strategies for mental calculation

Is mental calculation the same as mental arithmetic?

For many people, mental arithmetic is mainly about rapid recall of number facts – knowing your addition bonds to 20 and the tables to 10 × 10.

Rapid recall of number facts is one aspect of mental calculation but there is another. This involves challenging pupils with calculations where they have to figure out the answer rather than recall it from a bank of number facts that are committed to memory.

Examples
A pupil who knows that 9 + 9 = 18 can use this to calculate mentally other results, eg 9 + 8, 9 + 18, 19 + 9, 19 + 19.
A pupil who knows that 6 × 4 = 24 can use this fact to calculate 12 × 4 by doubling.
A pupil who knows that 36 ÷ 4 = 9 can use this to calculate 56 ÷ 4 by partitioning 56 into 36 + 20 and using the knowledge of 36 ÷ 4 and 20 ÷ 4 to reach the answer of 14.

Research shows that committing some number facts to memory helps pupils to develop calculation strategies, and that the use of strategies to figure out answers helps them commit further facts to memory. It is therefore important that schools focus on teaching mental strategies for calculation.

What's special about mental calculations?

Carrying out a calculation mentally is not the same as doing a traditional paper and pencil algorithm while mentally picturing it in your head.

Example
A group of pupils is carrying out mental addition of two-digit numbers.
Jo explains that 36 + 35 must be 71, since it is double 35 plus 1. She calls it 'a near double'.
Sam explains that 36 + 45 is 36 plus 40, making 76, plus 4, making 80, plus 1, making a total of 81. He has worked this out by partitioning the larger number and then bridging through 10.
Misha explains that 38 + 37 is 30 plus 30, or 60, plus 15, giving 75. She has partitioned both numbers into tens and units, and added the tens first. She has also recalled a known fact that she knows by heart: 8 + 7 = 15.

These explanations demonstrate a key feature of mental calculation: calculations that appear similar can be amenable to the use of different strategies, depending on the numbers involved.

This feature differs from a standard written method. Here, the strength of the method is that it is generally applicable to all cases irrespective of the numbers involved. For example, all the following calculations would be treated the same way if the standard pencil and paper algorithm for subtraction were used: 61 – 4, 61 – 41, 61 – 32, 61 – 58, 61 – 43. If carried out mentally, the numbers would provoke the use of different strategies.

There are clear advantages in expecting pupils to use mental calculation methods for calculations that might traditionally have been done with a standard paper and pencil algorithm. Such advantages include a stronger 'number sense', better understanding of place value, and greater confidence with numbers and the number system.

How do I help pupils to develop a range of mental strategies?

Individual pupils will be at different stages in terms of the number facts that they have committed to memory and the strategies available to them for figuring out other facts. Being aware of the range of possible strategies has the following advantages.

- When pupils are carrying out mental calculations, you will be in a better position to recognise the strategies being used.
- You can draw attention to and model a variety of strategies used by pupils in the class.
- You can make suggestions to pupils that will move them on to more efficient strategies.

There are three aspects to ensuring that pupils become effective in drawing on and using these strategies:

- raising pupils' awareness of the range of strategies
- developing their confidence and fluency with a range of strategies
- helping them to choose from the range the most efficient method for a given calculation.

Working on pupils' confidence and fluency with a range of strategies

Much of this booklet gives guidance on teaching methods for particular calculation strategies. There are two general approaches that support the development of mental strategies and help pupils to become increasingly competent, confident and efficient.

- **Using related calculations.** Many of the target pupils treat each calculation as a new one. By selecting problems carefully, you can help pupils to appreciate that from the answer to one problem, other answers can be generated. For example:
  - if pupils know that 6 × 7 = 42, they should immediately recognise three other facts: 7 × 6 = 42, 42 ÷ 7 = 6 and 42 ÷ 6 = 7
  - they can apply their knowledge of place value and work out answers to 60 × 7, 70 × 6, 0.7 × 6, 700 × 60, 0.06 × 0.7, and so on, and to divisions such as 420 ÷ 7, 42 000 ÷ 600, 4.2 ÷ 7, 42 ÷ 0.6, 0.42 ÷ 0.7
  - calculations like 26 × 7 can be expanded to (20 × 7) + (6 × 7) from which the answer of
    140 + 42 = 182 can be readily obtained.

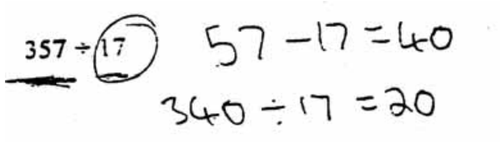
Revisiting this idea either in starters to a lesson, or even devoting a whole lesson to it from time to time, helps pupils to generate confidence in themselves and a feeling that they control calculations rather than calculations controlling them.

- **Talking through methods of solution.** Asking pupils to explain their methods of solution has the advantages that:
  - pupils doing the explaining clarify their own thinking
  - pupils who are listening develop their awareness of the range of possible methods
  - the activity can lead to a discussion of which method or methods are the most efficient.

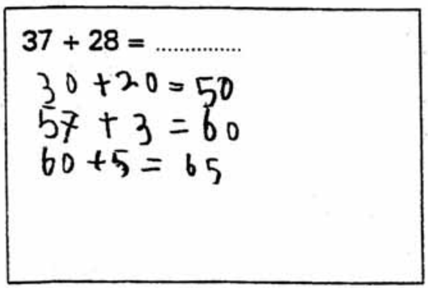
Can mental calculations involve paper and pencil?

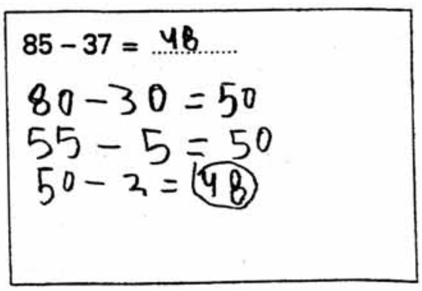
Paper and pencil can support mental calculation in various ways:

- through jottings – a record of the intermediate steps in a calculation

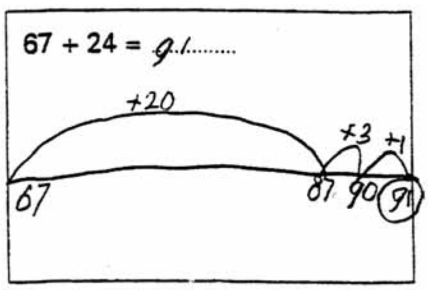


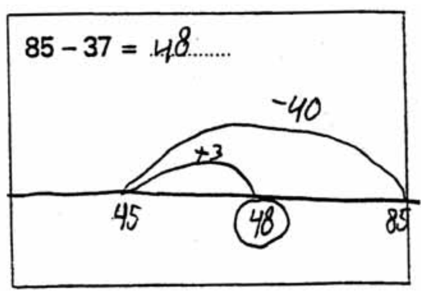
- through the idea of recording for another audience an explanation of the method used. This is a variation on getting pupils to explain their methods orally. A written account can help them begin to use appropriate notation and form the basis for developing more formal written methods





- through images – models and diagrams that support the development of mental imagery and that provide a visual representation of the way in which the calculation is being performed.



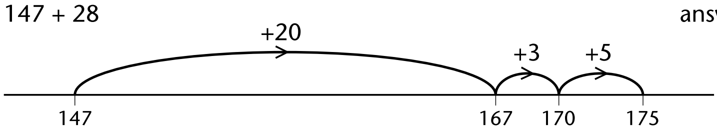


Using an empty number line

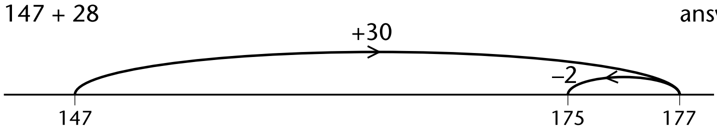
Research has shown that the empty (or blank) number line, as seen in the diagrams above, can be a powerful model for building understanding of and developing pupils' calculation strategies.

First, here are two examples of the empty number line representing addition.

147 + 28 answer = 175



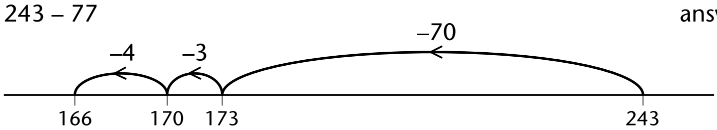
147 + 28 answer = 175



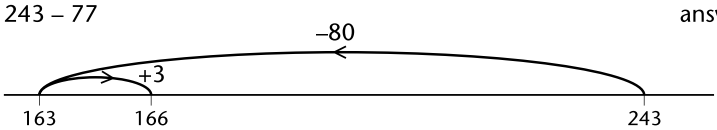
Both of these model common mental approaches. In each case, we mark the first or larger number. We then count on from that number by the amount we are adding on. In the second example, the fact that 28 is close to a decade number (30) prompts the compensation approach illustrated. The answer appears as a position on the number line.

One model of subtraction is the opposite of 'adding on' – subtraction is the inverse of addition. We count back from the first number by the amount we are subtracting.

243 – 77 answer = 166

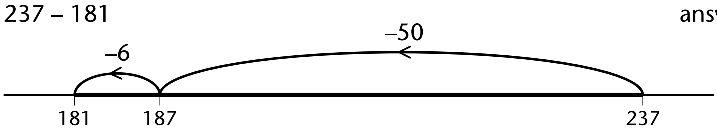


243 – 77 answer = 166

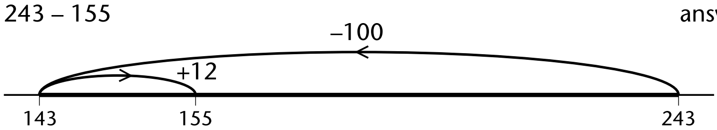


An alternative model for subtraction is to see it as the difference between two numbers. The difference is represented as the length of the line segment between the two numbers (the thicker lines in the examples below). This length can be calculated in different ways. Note that the answer is now the gap, not a position on the line.

237 – 181 answer = 56

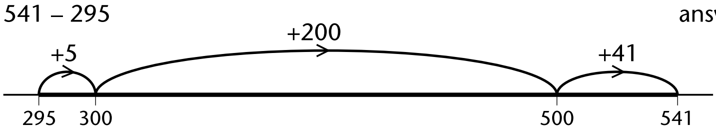


243 – 155 answer = 88



An alternative approach is to count up from the smaller to the larger number. This is analogous to a shop assistant counting out change.

541 – 295 answer = 246



Is speed important?

Once a strategy has been introduced to pupils, there comes a time to encourage them to speed up their responses and either use more efficient strategies or expand their repertoire. Rather than pitch the pupils against each other, it is better to encourage them to compete with themselves, aiming to better their previous performance. A prepared mental mathematics test can help pupils to monitor changes in performance over time.

The traditional model of a mental test is a set of unseen questions. An alternative is to give pupils examples of the type of questions 10 minutes in advance, so that they can think about the most efficient way to answer the questions. The purpose of this preparation time is not to try to commit answers to memory. Pupils should sort the questions into those they 'know' the answer to and those they need to figure out. Pairs of pupils can talk about their figuring-out methods and the whole class can spend some time discussing the strategies used. Collecting the questions, then giving pupils the test with the questions in a random order, also encourages attention to strategies. The same test can be used at a different time for pupils to try to beat their previous score.

One-minute multiplication test

Warn pupils in advance that they are going to be given a mental multiplication test.

Classes or individuals can be set different challenges according to levels of attainment, so one class or pupil may be working on multiples of 7 while another is working on multiples of 3. The same test paper can be duplicated for particular groups: ten randomly ordered 'multiplied by' questions:

$\square \times 3 = \square$

$\square \times 7 = \square$

$\square \times 4 = \square$

and so on.

Pupils write their particular multiple in the first box in each calculation. They then have one minute to fill in the answer boxes. Can they beat their previous best scores?

Other activities that encourage quick responses include the following.

Round the world: an activity to do with a group

Invite one pupil in a group to stand behind their neighbour's chair. Pose a mental mathematics question. Each of the two pupils tries to be first to answer. If the standing pupil correctly answers first, they move on to stand behind the next person. If the sitting pupil answers first, the two swap places and the pupil now standing up moves on to stand behind the next person. Continue asking questions until a pupil is back to the starting point. Who moved most places? Can anyone get 'round the world', that is, back to their original chair?

Beat the calculator

Invite a pair of pupils to the front and give one of them a calculator. Pose a calculation that you expect the pupils to be able to do mentally. The pupil with the calculator must use it to find the answer, even if they can mentally work more quickly. The best score out of five questions wins. You may have to be a strong referee to ensure that the calculator pupil doesn't call out the answer without keying in the calculation!

How should pupils respond to oral questions?

The traditional method of asking a question and inviting a volley of hands to go up has several drawbacks for pupils who are figuring out answers.

- It emphasises the rapid, the known, over the derived – pupils who know are the first to answer.
- Pupils who are collecting their thoughts are distracted by others straining to raise hands.

Ways of getting round these problems are:

- insisting that nobody puts a hand up until the signal, and silently counting to 5 or so before giving it
- using digit cards for all pupils to show their answer at the same time
- asking pupils to respond by writing their answers on mini-whiteboards.

Whatever method of response you use, allow time to discuss the various ways that pupils reached the answer, to point out the range of possible strategies and to highlight the most efficient and appropriate of them. Reinforce this by inviting individual pupils to the board to show their jottings and give an oral explanation of their method or strategy.

4 Addition and subtraction

This part of the booklet includes the following strategies:

- counting forwards and backwards
- reordering
- partitioning 1: using multiples of 10 and 100
- partitioning 2: bridging through multiples of 10
- partitioning 3: compensating
- partitioning 4: using near doubles
- partitioning 5: bridging through numbers other than 10.

Each section contains activities for use with pupils.

Features of addition and subtraction

The commutative law applies to addition but not subtraction. Numbers can be added in any order. Take any pair of numbers, say 7 and 12, then:

$$7 + 12 = 12 + 7$$

Order does matter in subtraction. Thus, 5 – 3 is not the same as 3 – 5. But a series of subtractions can be taken in any order. For example:

$$15 - 3 - 5 = 15 - 5 - 3$$

When three numbers are added together, they too can be taken in any order because of the associative law and/or the commutative law. In practice, two of the numbers have to be added together or associated first, and then the third number is added to the associated pair to give the result of the calculation. For example:

$$\begin{aligned} 7 + 5 + 3 &= (7 + 5) + 3 \\ &= 7 + (5 + 3) \\ &= (3 + 7) + 5 \end{aligned}$$

Because of the inverse relationship between addition and subtraction, every addition calculation can be replaced by an equivalent subtraction calculation and vice versa. For example, the addition:

$$\begin{aligned} 5 + 7 &= 12 \\ \text{implies } 5 &= 12 - 7 \\ \text{and } 7 &= 12 - 5 \end{aligned}$$

In the same way:

$$\begin{aligned} 13 - 6 &= 7 \\ \text{implies } 13 &= 7 + 6 \\ \text{and } 6 &= 13 - 7 \end{aligned}$$

Any numerical equivalence can be read from left to right or from right to left, so 6 + 3 = 9 is no different from 9 = 6 + 3.

Knowing addition and subtraction facts and counting forwards and backwards

Most pupils transferring into key stage 3 will probably be accustomed to starting at different numbers and counting forwards and backwards in steps, not only of ones, but also of twos, tens, hundreds and so on. The image of a number line helps them to appreciate the idea of counting forwards and backwards. They will also learn that, when adding two numbers together, it is generally easier to count on from the larger number rather than the smaller. You will need to review their counting on strategies, then show them and encourage them to adopt more efficient methods. You will also need to ensure that they commit certain facts to memory and can recall them rapidly.

Expectations leading up to national curriculum level 5

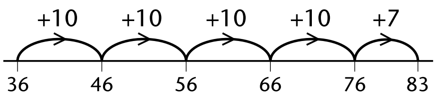
40 + 30	count on in tens from 40
6 + 8, 14 – 6	recall rapidly addition and subtraction facts to 20
60 + 40	recall rapidly all pairs of multiples of 10 with a total of 100
90 – 40	count back in tens from 90 or count on in tens from 40
65 + 35	recall rapidly all pairs of multiples of 5 with a total of 100
35 – 15	count on in steps of 3, 4 or 5 to at least 50
73 – 68	count on 2 to 70 then 3 to 73
86 – 30	count back in tens from 86 or count on in tens from 30
600 + 400	recall rapidly all pairs of multiples of 100 with a total of 1000
570 + 300	count on in hundreds from 300
960 – 500	count back in hundreds from 960 or count on in hundreds from 500
1½ + ¾	count on in quarters
1.7 + 0.5	count on in tenths

Activities

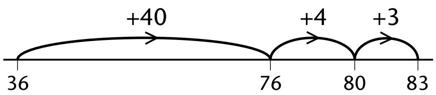
Get the pupils to count forward from zero in ones, one after the other round the class. When you clap your hands, they have to start counting backwards. On the next clap, they count forwards, and so on.

Extend to counting in tens, twos, threes, and other multiples. Vary the starting number. Include negative numbers.

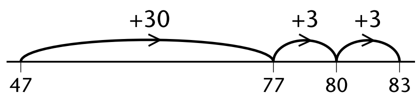
Get pupils to record a two-digit number on an empty number line. For example, 36 + 47 might be seen as counting on from 36 initially in steps of 10:



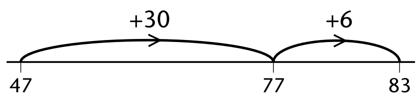
or first counting on in a step of 40:



or by reordering the calculation and then counting on from 47:



or:



Empty (or blank) number lines give a useful way for pupils to record their working and help you to see what method they are using. Discuss the different methods that pupils use. Encourage them to move to a more efficient method, getting them to compare the number of steps involved and the time taken to calculate the answer.

Make a number line that goes up in tens, large enough for the whole class or a group to see. A chalkboard, whiteboard or overhead projector would be appropriate.

Ask an individual pupil to show where a given number, such as 26, would fit on the line. Ask other pupils to fit some numbers close to 26, such as 23 or 28. They may find that the original position of the number 26 needs to be adjusted. Invite them to adjust the position of each number until they are satisfied with them. Then get them to explain what they did to the rest of the group or class.

This activity encourages pupils to imagine where the numbers 1 to 9, 11 to 19 and 21 to 29 would appear on the line, and to count on mentally before they decide where to place the number they are given. The idea can be extended to decimals, for example, with a line numbered 1, 2, 3 for positioning of numbers in tenths, or numbered 0.1, 0.2, 0.3 for positioning of numbers in hundredths.

Tell the class that you are going to walk along an imaginary number line. You will tell them what number you are standing on and what size steps you are taking.

For example: ‘I am on 15 and am taking steps of 10.’ Invite the class to visualise the number 15 on a number line and to tell you where you will be if you take one step forward (25). Take three more steps forward and ask: ‘Where am I now?’ (55). Take two steps back and ask: ‘Where am I now?’ (35), and so on.

Activities such as this help pupils to visualise counting on or back. The activity can be used for larger numbers. For example, tell pupils you are standing on 1570 and taking steps of 100 and ask them to visualise this. Then ask questions such as: ‘Where am I if I take two steps forward?’

Reordering

Sometimes a calculation can be more easily worked out by changing the order of the numbers. The way in which pupils rearrange numbers in a particular calculation will depend on which number facts they have instantly available to them.

It is important for pupils to know when numbers can be reordered:

eg 2 + 5 + 8 = 8 + 2 + 5   or   15 + 8 – 5 = 15 – 5 + 8   or   23 – 9 – 3 = 23 – 3 – 9

and when they can't be reordered:

eg 8 – 5 ≠ 5 – 8

The strategy of changing the order of numbers applies only when the question is written down. It is difficult to reorder numbers if the question is presented orally.

Expectations leading up to national curriculum level 5

23 + 54	= 54 + 23
12 – 7 – 2	= 12 – 2 – 7
13 + 21 + 13	= 13 + 13 + 21 (using double 13)
6 + 13 + 4 + 3	= 6 + 4 + 13 + 3
17 + 9 – 7	= 17 – 7 + 9
28 + 75	= 75 + 28 (thinking of 28 as 25 + 3)
3 + 8 + 7 + 6 + 2	= 3 + 7 + 8 + 2 + 6
25 + 36 + 75	= 25 + 75 + 36
58 + 47 – 38	= 58 – 38 + 47
200 + 567	= 567 + 200
1.7 + 2.8 + 0.3	= 1.7 + 0.3 + 2.8
34 + 27 + 46	= 34 + 46 + 27
180 + 650	= 650 + 180 (thinking of 180 as 150 + 30)
4.6 + 3.8 + 2.4	= 4.6 + 2.4 + 3.8
8.7 + 5.6 – 6.7	= 8.7 – 6.7 + 5.6
4.8 + 2.5 – 1.8	= 4.8 – 1.8 + 2.5

Activities

Present pupils with groups of four numbers that they are to add in their head. Make sure that, in each group of numbers, there are two numbers that have a total of 10, for example:

8 + 3 + 5 + 2

Discuss their methods. See if any pupils chose to add 8 + 2 first and then add on the 5 + 3, or linked the 3 + 5 and added 8 + (3 + 5) + 2.

Give pupils similar examples and encourage them to look for pairs that add to make 10 or that make doubles before they start to add. Get them to make up similar examples for each other.

Have regular short, brisk practice sessions where pupils are given ten questions such as:

2 + 7 + 8 + 5 + 4 + 3

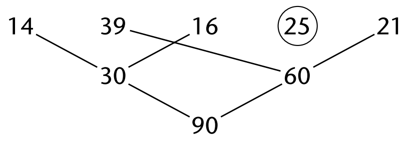
with at least five numbers, such that some pairs total 10. Encourage pupils to time their responses, keep a personal record of their times and try to beat their personal best.

Give pupils the same set of questions at regular intervals and encourage them to see how rapidly they can get to the answers. This should ensure that every pupil sees that they have made progress.

When pupils can find pairs of numbers that add to make multiples of 10, they can make use of this information when adding several numbers together. For example, when adding:

14 + 39 + 16 + 25 + 21

it is sensible to pair numbers:



90 + 25 = 115

Pupils should learn that it is worth looking at all numbers that are to be added to see whether there are pairs that make convenient multiples of 10. The number tree shown in the diagram can be a helpful model of the ways the numbers were paired.

In some particular sequences of numbers, the reordering strategy is useful and can give opportunities for an investigative approach. For example:

Find quick ways of finding these answers:

1 + 2 + 3 + 4 + 5 + 6 = ?

5 + 7 + 11 + 13 = ?

3 + 6 + 9 + 12 + 15 + 18 = ?

1 + 2 + 3 + 4 + … + 98 + 99 = ?

Series of numbers such as these are always easier to add by matching numbers in pairs. In the first, it is easier to add 1 + 6 = 7, 2 + 5 = 7, 3 + 4 = 7, and then to find 7 × 3. In the last example, combining 1 + 99, 2 + 98 and so on up to 49 + 51, gives the total (100 × 49) + 50, or 4950.

Use a set of number cards, making sure that there are pairs that make multiples of 10. Divide the class into groups of three and give each pupil a card. Ask each group to add their numbers together. Encourage them to look for pairs of numbers to link together. List all the totals on a board or projector. Ask whose numbers give the largest total. Get pupils to swap their cards with someone in another group and repeat.

The numbers on the cards can depend on the knowledge of the pupils. For example, they could be:

23	30	17	52	24	8	70	16
12	30	60	140	170	50	80	

You can arrange for each group to get cards that match their number skills.

The activity can be extended to using decimals, but in this case the aim is to make pairs that make a whole number: eg use 1.4, 3.2, 0.6, 0.2, 1.6, 0.8, 2.3.

Partitioning 1: using multiples of 10 and 100

It is important for pupils to know that numbers can be partitioned into, for example, hundreds, tens and ones, so that 326 = 300 + 20 + 6. In this way, numbers are seen as wholes, rather than as a collection of single digits in columns. This way of partitioning numbers can be a useful strategy for addition and subtraction. Both of the numbers involved can be partitioned in this way, although it is often helpful to keep the first number as it is and to partition just the second number.

Expectations leading up to national curriculum level 5

30 + 47	= 30 + 40 + 7
78 – 40	= 70 – 40 + 8
25 + 14	= 20 + 5 + 10 + 4 = 20 + 10 + 5 + 4
23 + 45	= 40 + 5 + 20 + 3 = 40 + 20 + 5 + 3
68 – 32	= 60 + 8 – 30 – 2 = 60 – 30 + 8 – 2
55 + 37	= 55 + 30 + 7 = 85 + 7
365 – 40	= 300 + 60 + 5 – 40 = 300 + 60 – 40 + 5
43 + 28 + 51	= 40 + 3 + 20 + 8 + 50 + 1 = 40 + 20 + 50 + 3 + 8 + 1
5.6 + 3.7	= 5.6 + 3 + 0.7 = 8.6 + 0.7
4.7 – 3.5	= 4.7 – 3 – 0.5
540 + 280	= 540 + 200 + 80
276 – 153	= 276 – 100 – 50 – 3

Activities

Use a dice marked 1, 1, 10, 10, 100, 100 for the game ‘target 500’ with a group of pupils. Each player may roll the dice as many times as they wish, adding the score from each roll and aiming at the target of 500. They must not overshoot. If they do, they go bust!

For example, a sequence of rolls may be:

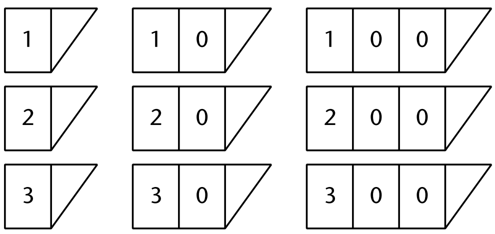
10, 10, 1, 100, 1, 100, 100, 1, 1, 10, 100

At this point, with a total of 434, a player might decide not to risk another roll (in case 100 is rolled) and stop, or to hope for another 10.

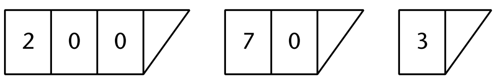
The winner is the player who gets nearest to 500.

This game practises building up numbers by mental addition using ones, tens and hundreds.

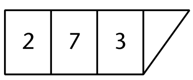
Use place value cards 1 to 9, 10 to 90 and 100 to 900:



Ask pupils to use the cards to make a two-digit or a three-digit number by selecting the cards and placing them on top of each other. For example, to make 273, the cards



can be placed over each other to make



This could be used as a class activity, in which case you could ask individual pupils to select the appropriate card and to put it in the correct place. Alternatively, it could be used with individuals as a diagnostic task, to check whether they understand place value in this context.

With a group of three to five players, play a game using the place value cards 1 to 9, 10 to 90 and 100 to 900. You also need a set of two- and three-digit number cards to use as target numbers, for example:

153	307	682	914	530	890
745	201	26	79	468	96

Put the target numbers in a pile and turn them over, one by one, to set a target. Deal the 27 place value cards between the players.

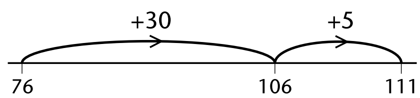
Player A inspects their cards to see if they have any part of the target number. If so, they put it on the table.

Play continues anticlockwise. Player B checks to see whether they have another part of the number, followed by players C, D, and so on. Whoever completes the target number keeps it. The winner is the player who wins the most target numbers.

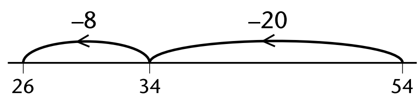
Games like this give motivating practice in partitioning numbers into hundreds, tens and ones.

Use the empty number line to add or subtract two-digit numbers, for example:

76 + 35:



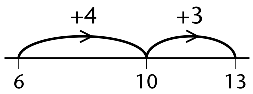
or 54 – 28:



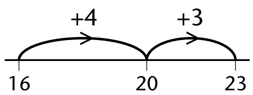
Empty number lines are a useful way to record how pupils use multiples of 10 or 100 to add or subtract. They allow pupils to discuss different methods and encourage the use of more efficient methods.

Partitioning 2: bridging through multiples of 10

An important aspect of having an appreciation of number is to know when a number is close to 10 or a multiple of 10: to recognise, for example, that 47 is 3 away from 50, or that 96 is 4 away from 100. When adding or subtracting mentally, it is often useful to make use of the fact that one of the numbers is close to 10 or a multiple of 10 by partitioning another number to give the difference. The use of an empty number line where the multiples of 10 are seen as landmarks is helpful and enables pupils to have an image of jumping forwards or backwards to these landmarks. For example:



In the case of subtraction, bridging through the next 10 or multiple of 10 is a very useful method (often called 'shopkeeper's subtraction', it is the method used almost universally when giving change with money). So the change from £1 for a purchase of 37p is carried out thus: '37 and 3 is 40, and 10 is 50, and 50 is £1.' The use of actual coins, or the image of coins, helps to keep track of the subtraction. The empty number line can give an image for this method when the subtraction does not involve money. The calculation 23 – 16 can be built up as an addition:



'16 and 4 is 20, and 3 is 23, so add 4 + 3 for the answer.'

A similar method can be applied to the addition and subtraction of decimals, but here, instead of building up to a multiple of 10, numbers are built up to a whole number or to a tenth. So 2.8 + 1.6 can be turned into 2.8 + 0.2 + 1.4 = 3 + 1.4.

Expectations leading up to national curriculum level 5

6 + 7	= 6 + 4 + 3
23 – 9	= 23 – 3 – 6
15 + 7	= 15 + 5 + 2
49 + 32	= 49 + 1 + 31
57 + 44	= 57 + 3 + 41
3.8 + 2.6	= 3.8 + 0.2 + 2.4
5.6 + 3.5	= 5.6 + 0.4 + 3.1
296 + 134	= 296 + 4 + 130
584 – 176	= 584 – 184 + 8
0.8 + 0.35	= 0.8 + 0.2 + 0.15

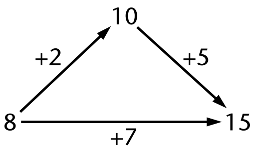
Activities

Show the class a single-digit number and ask a pupil to find its complement in 10. Repeat this many times, encouraging pupils to respond as quickly as they can. Then offer two-digit numbers and ask for complements to 100.

Activities such as these give practice so that pupils can acquire rapid recall of complements to 10 or 100.

Give the class two single-digit numbers to add. Starting with the first number (the larger), ask what part of the second number needs to be added to make 10, then how much more of it remains to be added on. Show this on a diagram like this:

8 + 7 = 8 + 2 + 5 = 10 + 5 = 15



Diagrams like this are a useful model for recording how the starting number is built up to 10 and what remains to be added. Pupils can be given blank diagrams and asked to use them to add sets of numbers less than 10.

Give examples based on money, asking pupils to show what coins they would use to build up to the next convenient amount. For example:

A packet of crisps costs 27p. How much change do you get from 50p?

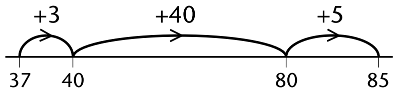
27p and 3p is 30p. Another 20p makes it up to 50p. The change is 3p + 20p = 23p.

This is the natural way to find a difference when using money. The coins provide a record of how the change was given. This method can usefully be applied to other instances of subtraction.

Use empty number lines for addition and subtraction, using multiples of 10 as interim numbers. For example:

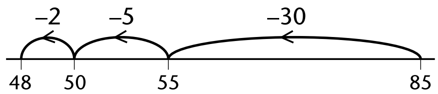
85 – 37

(i) by adding on from 37 to 85 (the shopkeeper's method):



'37 and 3 makes 40, and 40 make 80, and 5 makes 85. So add 3 + 40 + 5 to get the answer.'

(ii) by counting backwards:



In the first example, the next multiple of 10 is a first landmark from the starting number, so 37 is built up to 40. Subsequent landmarks might be other multiples of 10 (80 in the first case). In the second example, the first landmark is 55, then the next is 50. Encouraging pupils to use number lines in this way provides a mental image that can assist with mental calculations.

Write a set of decimal numbers on the board, such as:

3.6, 1.7, 2.4, 6.5, 2.3, 1.1, 1.5, 1.8, 2.2, 3.9

Ask pupils to find pairs that make a whole number.

Extend the activity to finding pairs of numbers that make a whole number of tenths:

0.07, 0.06, 0.03, 0.05, 0.04, 0.05, 0.09, 0.01

The first activity with decimals is to build up to whole numbers, so 3.6 is added to 2.4 to make 6. In the case of hundredths, pairs of numbers such as 0.06 and 0.04 can be added to make 0.1.

Partitioning 3: compensating

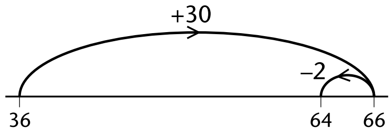
This strategy is useful for adding and subtracting numbers that are close to a multiple of 10, such as numbers that end in 1 or 2, or 8 or 9. The number to be added or subtracted is rounded to a multiple of 10 plus or minus a small number. For example, adding 9 is carried out by adding 10, then subtracting 1; subtracting 18 is carried out by subtracting 20, then adding 2. A similar strategy works for decimals, where numbers are close to whole numbers or a whole number of tenths. For example, 1.4 + 2.9 = 1.4 + 3 – 0.1 or 2.45 – 1.9 = 2.45 – 2 + 0.1.

Expectations leading up to national curriculum level 5

34 + 9	= 34 + 10 – 1
52 + 21	= 52 + 20 + 1
70 – 9	= 70 – 10 + 1
53 + 11	= 53 + 10 + 1
58 + 71	= 58 + 70 + 1
84 – 19	= 84 – 20 + 1
38 + 69	= 38 + 70 – 1
53 + 29	= 53 + 30 – 1
64 – 19	= 64 – 20 + 1
138 + 69	= 138 + 70 – 1
405 – 399	= 405 – 400 + 1
2½ + 1¾	= 2½ + 2 – ¼
5.7 + 3.9	= 5.7 + 4.0 – 0.1

Activities

Use a number line to carry out additions such as 36 + 28 by counting on 30 and compensating by counting back by 2:



Ask pupils to visualise a number line to show 45 + 29, 27 + 39, …

The number line is a means of showing how the process of counting on and then back works. It can also be a useful way of getting pupils to visualise similar examples when working mentally.

Prepare a set of cards with numbers such as 3.9, 2.9, 5.1, 4.1, 5.8, 3.2 and so on and work with a group of pupils.

3.9	2.9	5.1	4.1	5.8	3.2

Pupils take turns to choose a single-digit number, to turn over one of the prepared cards and then add the two numbers. Get pupils to tell others their addition strategy. Encourage them to see, for example:
7 + 4.1 as 7 + 4 + 0.1 and 7 + 3.9 as 7 + 4 – 0.1.

Extend this to decimals by choosing a number with one decimal place as the starting number. So, for example, 2.4 + 3.9 = 2.4 + 4 – 0.1.

Practise examples such as 264 – 50, 3450 – 300. Then ask pupils to subtract numbers such as 9, 99, 999 or 11, 31, 91 and so on.

Encourage them to use a method of rounding and compensating, so 264 – 39 = 264 – 40 + 1 and 2500 – 99 = 2500 – 100 + 1.

This activity could be a class activity. Individual pupils round the class could give the answer. Encourage explanation of strategies.

Use a number square for adding tens and numbers close to 10. To find 36 + 28, first find 36 + 30 by going down three rows, then compensate by going back along that row two places:

31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70

Pupils need to know that they can use the table to add 10 to any number by moving down to the number below it. For example, 36 + 10 = 46, which is just below 36, and 36 + 20 is 56, to be found two rows below. Subtracting 10 is modelled by moving to numbers in the row above. Pupils can use this strategy for adding or subtracting numbers that are close to a multiple of 10 by finding the correct row and then moving to the right or the left.

Prepare two sets of cards for a subtraction game. Set A has numbers from 12 to 27. Set B contains only 9 and 11 so that the game involves subtracting 9 and 11. Shuffle the cards and place them face down.

Each pupil needs a playing board.

15	3	9	16
5	18	4	17
13	7	12	8
6	11	14	10
18	1	2	17

Pupils take turns to choose a number from set A and then one from set B. They subtract the number from set B from the one from set A and mark the answer on their board. The first person to get three numbers in a row on their board wins.

Discuss the strategy used to subtract. Encourage the use of compensating.

By changing the numbers in set A, negative numbers can also be used.

Partitioning 4: using near doubles

If pupils have instant recall of doubles, they can use this information when adding two numbers that are very close to each other. So, knowing that 6 + 6 = 12, they can be encouraged to use this to help them find 7 + 6, rather than use a 'counting on' strategy or 'building up to 10'.

Expectations leading up to national curriculum level 5

18 + 16	is double 18 and subtract 2 or double 16 and add 2
36 + 35	is double 36 and subtract 1 or double 35 and add 1
60 + 70	is double 60 and add 10 or double 70 and subtract 10
38 + 35	is double 35 and add 3 or double 38 and subtract 3
160 + 170	is double 150, then add 10, then add 20 or double 160 and add 10 or double 170 and subtract 10
380 + 380	is double 400 and subtract 20 twice
1.5 + 1.6	is double 1.5 and add 0.1 or double 1.6 and subtract 0.1
421 + 387	is double 400 add 21, and then subtract 13

Activities

Choose a 'double fact' and display it on the board, for example:

8 + 8 = 16

Invite someone to give the total. Then ask for suggestions of addition facts that pupils can make by changing one of the numbers, for example:

8 + 9 = 17   7 + 8 = 15

Repeat the activity by giving pupils double facts that they might not know, such as:

17 + 17 = 34   28 + 28 = 56   136 + 136 = 272

Then ask pupils to say how they could work out:

17 + 18 or 16 + 17 or 27 + 28 or 136 + 137

Invite pupils to give their own double fact and ask other pupils to suggest some addition facts that they can generate from it. You can also extend this activity to decimals.

Working with a whole class, ask one pupil to choose a number less than 10. Then ask pupils, in turn, to double the number, then double the result and so on. Get them to write the numbers on the board or on an overhead projector, for example:

3 6 12 24 48 96 …

Challenge them to see how far they can go with this doubling sequence.

Ask them to produce another sequence by starting with a number, then doubling it and adding 1 each time. So starting again with 3, say, they would get:

3 7 15 31 63 …

Variations would be to use the rules 'double, then subtract 1' or 'double, then add 2'.

You could try this investigation as a class activity. Use these rules: 'If a number is odd, multiply it by 3 and add 1. If it is even, halve it.' Try some different starting numbers. What happens? What is the longest chain you can get?

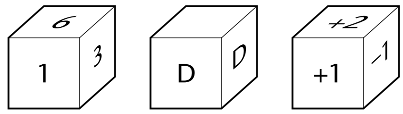
Being proficient at doubling is essential if pupils are to find near doubles. Asking pupils to produce the next term in a sequence ensures that they must all take note of answers given by other pupils.

Play 'think of a number'. Use a rule that involves doubling and adding or subtracting a small number, for example:

'I'm thinking of a number.
I double it and add 3.
My answer is 43.
What was my number?'

'Think of a number' activities require pupils to 'undo' a process by using inverse operations. This activity gives practice in both halving and doubling. Invite pupils to invent similar examples themselves.

This game for a group of pupils needs three dice. One dice is numbered 1 to 6; a second has four faces marked with a D for 'double' and two blank faces; the third is marked +1, +1, +1, +2, –1, –1.



Pupils take turns to throw the three dice and record the outcome. They then decide what number to make. For example, if they throw 3, D and +1 they could double the 3 then add 1 to make 7, or they could add 1 to the 3 to make 4 and then double 4 to make 8. Ask them:

'What is the smallest possible total?'
'What is the biggest?'
'What totals are possible with these three dice?'
'Which totals can be made the most ways?'

Games such as this motivate pupils to practise the strategy and consider questions such as: 'What totals are possible?' Encourage pupils to reflect on the processes rather than to find just one answer.

Get pupils to practise adding consecutive numbers such as 45 and 46. Then give pupils statements such as: 'I add two consecutive numbers and the total is 11.' Ask them: 'What numbers did I add?'

Knowing doubles of numbers is useful for finding the sum of consecutive numbers. The reverse process is more demanding.

Partitioning 5: bridging through numbers other than 10

Time is a universal non-metric measure. A digital clock displaying 9:59 will, two minutes later, read 10:01 not 9:61. When working with minutes and hours, it is necessary to bridge through 60. Hours and days require bridging through 24. So to find the time 20 minutes after 8:50, for example, pupils might say: '8:50 + 10 minutes takes us to 9:00, then add another 10 minutes'.

Expectations leading up to national curriculum level 5

1 minute = 60 seconds
1 hour = 60 minutes
1 day = 24 hours
1 week = 7 days
1 year = 12 months = 365 days (or 366 days in a leap year)
It is half past seven. What time was it 3 hours ago?
It is 7 o'clock in the morning. How many hours to midnight?
It is 10:30. How many minutes to 10:45?
It is 3:45. How many minutes to 4:15?
What time will it be 40 minutes after 3:30?
What is the time 50 minutes before 1:00 pm?
It is 4:25. How many minutes to 5:00?
What time will it be 26 minutes after 3:30?
What is the time 33 minutes before 1:00 pm?
It is 4:18. How many minutes to 5:00?
It is 08:35. How many minutes is it to 09:15?
It is 11:45. How many hours and minutes is it to 15:20?
A train leaves London for Leeds at 22:33. The journey takes 2 hours 47 minutes. What time does it arrive?

Activities

Have a digital clock in the classroom.

Get the class to look at it at various times of the day and ask: ‘How many minutes it is to the next hour (or next o’clock)?’

Encourage pupils to count on from 36 to 40, then to 50, then to 60, to give a total of 24 minutes.

Then ask questions such as: ‘How long will it be to 11:15?’ Get them to count on to 11:00 and then add on the extra 15 minutes.

The calculation can be modelled on a number line labelled in hours and minutes.

Some pupils may think that minutes on digital clocks behave like ordinary numbers, so that they might count on 59, 60, 61 and so on, not realising that at 60 the numbers revert to zero as the hour is reached. It helps if you draw attention to what happens to the clock soon after, say, 9:58 and to stress the difference between this and other digital meters such as electricity meters or the meters that give the distance travelled by a bicycle or car.

Give the class, or a group of pupils, statements such as:

‘Jane leaves home at 8:35 am. She arrives at school at 9:10 am. How long is her journey?’

Discuss how to find the answer and discuss their various methods, writing each on the board. Some might say:

‘8:35, 8:40, 8:50, 9:00, 9:10’

counting 5 and 10 and 10 and 10 to give the total time. Others might say:

‘8:35 and 25 minutes takes us to 9:00, so add on another 10 minutes.’

Pupils need to remember that, for minutes, they need to count up to 60 before getting to the next hour. Some pupils might be tempted to say ‘8:35, 8:40, 8:50, 8:60’, and so on, expecting to go on until they get to 100. Referring to a clockface should help them to see why this is incorrect.

Use local bus or train timetables.

This is part of a train timetable from Bristol to London:

Bristol	**London**
08:30	09:45
10:15	11:40
12:30	13:55
15:15	16:48

Ask questions such as: ‘How long does the 8:30 train take to get to London?’ Encourage pupils to count up to 9:00 and then to add on the extra 45 minutes.

Ask: ‘Which train takes the shortest time? Which takes the longest?’

Suggest that pupils build the starting times up to the next hour, and then add on the remaining minutes.

Plan a journey using information from a timetable. For example, coaches arrive at and leave Alton Towers at these times:

	Arrive	**Leave**
Coach A	08:00	14:30
Coach B	09:30	15:45
Coach C	10:15	16:00
Coach D	11:45	17:30

Ask questions such as: ‘Which coach gives you the most time at Alton Towers? Which gives you the least?’

Discuss the strategies that pupils use to find the times. For coach C, for example, some might bridge 10:15 up to 11:00 and then find the number of remaining hours; others might bridge from 10:15 through 12:15 to 15:15, counting in hours and then add on the remaining 45 minutes. Pupils need to remember that they need to count the hours 10, 11, 12, and then start again with 1, 2, and so on. Again, an empty number line can be used to model the calculations.

This extract has been removed due to copyright issues.
Original publication DfES: 0744-2004

5 Multiplication and division

This part of the booklet includes the following strategies:

- knowing multiplication and division facts to 10 × 10
- multiplying and dividing by multiples of 10
- multiplying and dividing by single-digit numbers and multiplying by two-digit numbers
- doubling and halving
- fractions, decimals and percentages.

Each section contains activities for use with pupils.

Features of multiplication and division

Three different ways of thinking about multiplication are:

- as repeated addition, for example, 3 + 3 + 3 + 3
- as an array, for example, four rows of three objects
- as a scaling factor, for example, making a line three times longer.

The use of the multiplication sign can cause difficulties. Strictly, 3 × 4 means 3 multiplied by 4 or 3 + 3 + 3 + 3, which is four lots of three, or four threes. This is counter to the intuitive way of interpreting 3 × 4, which is often thought of as 3 times 4, which is three lots of four, or three fours. Fortunately, multiplication is commutative: 3 × 4 is equal to 4 × 3, so the outcome is the same. The colloquial use of 'three times four' is somewhat confusing: it is a phrase that was derived, presumably, from the idea of three, taken four times – or four taken three times.

An expression such as (4 + 5) × 3 involves both multiplication and addition. The distributive law of multiplication over addition means that:

$(4 + 5) \times 3 = (4 \times 3) + (5 \times 3)$

This can be very useful in mental calculation.

Division and multiplication are inverse operations. For mental calculation, it is important to know multiplication facts in order that the related division facts can be worked out.

Because knowledge of multiplication and the corresponding division facts up to 10 × 10 is so important for pupils entering key stage 3, the first two sections in this part of the booklet concentrate on how pupils acquire these facts.

Knowing multiplication and division facts to 10 × 10

Instant recall of multiplication and division facts is a key objective in developing pupils' numeracy skills. The expectation is that most pupils entering key stage 3 will know the multiplication facts up to 10 × 10 and that many of them will also know the associated division facts.

However, learning these facts and being fluent at recalling them quickly is a gradual process that takes place over time. The target group of pupils, particularly those who have achieved level 3 in the key stage 2 tests, may still be struggling with remembering multiplication facts and have at best shaky knowledge of division facts.

Fluent recall of multiplication and division facts relies on regular opportunities for practice in a variety of situations. The ability to work out and knowing by heart are linked and support each other. For example, the pupil who can work out the answer to 8 × 6 by recalling 8 × 5 and then adding 8 will, through regular use of this strategy, become more familiar with the fact that 8 × 6 is 48.

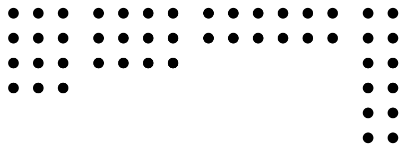
In the interest of speed and accuracy, it is important that pupils know multiplication and division facts by heart. Pupils need a great deal of practice if they are to do this. Generally, frequent short sessions are more effective than longer, less frequent sessions. It is crucial that the practice involves as wide a variety of activities as possible. Some suitable ones are suggested on the following pages.

Expectations leading up to national curriculum level 5

Count from and back to zero in twos, threes, fours, fives or tens
Recall the 2, 3, 4, 5 and 10 times tables, up to the tenth multiple
Recall the division facts related to the 2, 3, 4, 5 and 10 times tables
Find the remainder when a number to 20 is divided by 2, a number to 50 is divided by 5, or a two-digit number is divided by 10
Recognise odd and even numbers
Count on or back from any number in twos, threes, fours, fives and tens
Count from and back to zero in sixes, sevens, eights or nines
Recall the 6, 7, 8 and 9 times tables, up to the tenth multiple
Recall square numbers up to 10 × 10
Recall division facts related to the 6, 7, 8 and 9 times tables
Give the remainder when a number to 30 is divided by 3, a number to 40 is divided by 4, …, a number to 90 is divided by 9
Count on or back from any number in sixes, sevens, eights or nines
Recall the squares of 11 and 12 (ie 11 × 11 and 12 × 12)
Recall the square roots of perfect squares to 144
Know and use tests of divisibility for 2, 3, 4, 5, 9 and 10
Find the factors of numbers to 100
Identify prime numbers to 100

Activities

Discuss ways of grouping a number of dots in a rectangular array. For example, 12 can be represented as follows:



Pupils can use this idea to play this game.

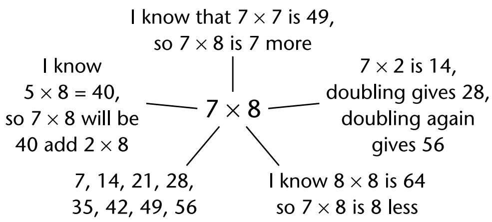
Player A takes a handful of counters, counts them and tells player B how many there are. Player B then says how the counters can be arranged in a rectangular array and proceeds to make it. (Single line arrays are not allowed.) If both players agree the array is correct, player B gets a point.

Both players record the two multiplication facts that the array represents. For example, a 5 by 3 array is recorded as 15 = 3 × 5 and 15 = 5 × 3, or 3 × 5 = 15 and 5 × 3 = 15.

After the game, discuss numbers that can't be made into a rectangular array, ie the primes.

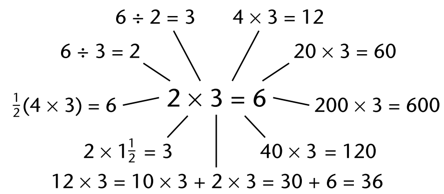
Arranging counters in a rectangular array is a helpful introduction to understanding about factors. If a number can be arranged in a rectangle (excluding a straight line) then it can be factorised. Numbers that can be arranged only as a straight line are primes.

Write a multiplication in the middle of the board. Invite pupils to the board to explain and record how they would figure out the result. Record different methods and use them as a basis for discussion.



Explaining to others consolidates a pupil's own understanding. It also allows other pupils to be introduced to methods that they might not have thought of for themselves.

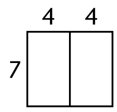
Write a multiplication fact in the middle of the board and ask pupils: 'Now that we know this fact, what other facts do we also know?' Invite pupils to the board to explain and record their ideas.



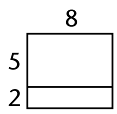
Again, pupils who are explaining to others clarify their own thoughts and others are introduced to different methods.

Use the empty rectangle to model multiplication and division facts that can be worked out from known facts. For example:

7 × 8 is 7 × 4 and another 7 × 4:



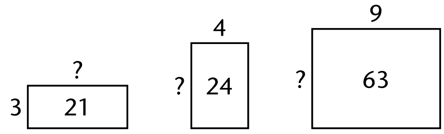
7 × 8 is 5 × 8 and 2 × 8:



Use the same method for bigger numbers, for example, 23 × 4 is 20 × 4 and 3 × 4.

These diagrams, which are based on areas, are useful later when studying multiplication in algebra. They show how the distributive law works with multiplication.

Use rectangles to practise division:



21 ÷ 3 = ?   24 ÷ 4 = ?   63 ÷ 9 = ?

The same model that was used in the previous example can be used for division, and helps to show that multiplication and division are inverse operations.

More activities…

Show pupils all the 10 × 10 multiplication facts displayed in a table.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Ask pupils to highlight the facts that they already know (eg multiplication by 1, 2, 5 and 10).

Then identify facts that they can work out easily, eg the 4 times table is twice the 2 times table, the 8 times table is twice the 4 times table, the 9 times table is one multiple less than the 10 times table (eg 6 × 9 is 6 less than 6 × 10, and so on).

Now identify the 'tricky' facts that they need to work on, for example 7 × 6, 8 × 9.

Multiplication squares display the patterns that exist in the tables. Pupils might like to keep their own personal square and to colour in facts as they are learned.

Play a game for two or three players. Each pupil throws a dice twice to obtain the length and width of a rectangle. They then draw the rectangle on squared paper, find its area and see whose rectangle is the largest one.

Ask questions such as:

- 'How did you work that out?'
- 'How did you split up the rectangle?'
- 'What multiplication have you worked out?'
- 'What division fact comes from that?'
- 'What if the number on that side of the rectangle were twice as big (or 10 more, or 10 times bigger, or ...)?'

This activity uses an area model for multiplication.

Distribute a number of cards with a multiplication fact with one number missing, such as:

? × 4 = 24   5 × ? = 35

Pupils need to place their cards on the space that gives the missing number on a sorting tray like this:

2	3	4
5	6	7
8	9	10

Pupils should time themselves to see how long it takes them to sort all nine cards correctly. Repeat several times over a month to see if they can improve on their time.

Place tracing paper over a random collection of numbers such as:

3	4	9
8	7	8
1	2	3
2	6	8
5	4	9
9	6	7

To practise recalling multiples of a given number, eg multiples of four, pupils write the appropriate multiple over each printed number.

12	16	36
32	28	32
4	8	12
8	24	32
20	16	36
36	24	28

How many can they do in two minutes?

This time pupils are given a fixed time to see how many multiplication facts they know. The activity can then be repeated several times over the next few days to see how their speed of recall improves.

...and more...

Design a set of cards containing questions and answers. Answers should not be to the question above them but they are answers to another question on a different card.

3×5	6×7	4×4	9×8	8×5
40	15	42	16	72

Distribute the cards around the class so that everyone has at least one (usually two or three). Ask one pupil to read out the multiplication at the top of their card. The pupil who has the correct answer reads it out and then reads out the question at the top of that card. This continues until all cards have been used up.

The game works best if the cards form a loop where the question on the last card is linked to the answer on the first, as in the five cards above.

One of the benefits of activities like this is that every pupil has to work out all the answers to see if they have the one that is asked for. This means that every pupil gets plenty of practice.

Make cards showing a multiplication on one side and the answer on the other. Pupils put the cards out in front of them with either all the multiplications or all the results showing.

1×7	7×9	4×7	9×7
5×7	6×7	2×7	6×7
7×6	3×7	10×7	7×5

A player touches a card, says what is on the other side and then turns it over. If not correct, the card is turned back over. Another card must then be tried. Play continues until all the cards are turned over.

Keep a record of the time taken to complete the activity and try to improve on it next time.

Show pupils this grid and explain that it contains the numbers that appear in the 2, 5 and 10 times tables. Some numbers appear in more than one table.

4	6	8
10	12	14
15	16	18
20	25	30
35	40	45
50	60	70
80	90	100

Point to a number and invite pupils to say what the multiplication fact is. Encourage a quick response.

Do the same activity with the 3, 4, 6, 7, 8 and 9 multiplication tables, using this chart:

9	12	16	18
21	24	27	28
32	36	42	48
49	54	56	63
64	72	81	

This activity requires pupils to recognise the product of two numbers and to say what those numbers are.

Ask pupils:

'How did you get the answer 42?'

'What two numbers did you multiply to get that answer?'

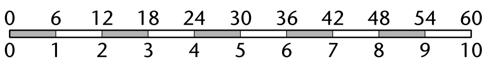
This activity shows up the inverse relationship between multiplication and division.

Chant division tables, using 'divided by':

$0 \div 6 = 0$
$6 \div 6 = 1$
$12 \div 6 = 2$

and so on.

This can be supported with a counting stick. The stick helps to establish the relationship between the increasing steps and the corresponding quotients.



Using a vertical counting stick makes a more direct correspondence to a recorded division table.

Multiplying and dividing by multiples of 10

Being able to multiply by 10 and multiples of 10 depends on an understanding of place value. This ability is fundamental to being able to multiply and divide larger numbers.

Expectations leading up to national curriculum level 5

7 × 10, 70 ÷ 10
26 × 10, 260 ÷ 10
6 × 100, 600 ÷ 100
35 × 100, 3500 ÷ 100
Change 80 mm to cm; change 25 cm to mm
Write 45 metres in centimetres; write 500 cm in metres
672 × 10, 6720 ÷ 10
935 × 100, 93 500 ÷ 100
5231 × 10, 52 310 ÷ 10
1468 × 100, 146 800 ÷ 100
7 × 30 = 7 × 3 × 10
42 × 200 = 42 × 2 × 100
360 ÷ 90 = 360 ÷ 10 ÷ 9
4200 ÷ 600 = 4200 ÷ 100 ÷ 6
Change 5 hours to minutes, and 20 minutes to seconds, and vice versa
5 × 1000, 5000 ÷ 1000
31 × 1000, 31 000 ÷ 1000
216 × 1000, 216 000 ÷ 1000
4391 × 1000, 4 391 000 ÷ 1000
4.7 × 10, 47 ÷ 10
3.6 × 100, 360 ÷ 100
1.4 × 1000, 1400 ÷ 1000
5.32 × 10, 53.2 ÷ 10
6.95 × 100, 695 ÷ 100
4.78 × 1000, 4780 ÷ 1000
Change 8.3 m to cm; change 420 cm to metres
Change 5.75 kg to g; change 6950 g to kg
6 × 0.1, 0.6 ÷ 0.1
7 × 0.01, 0.07 ÷ 0.01
31 × 0.4 = 31 × 4 × 0.1
81 ÷ 0.3 = 81 ÷ 3 ÷ 0.1

Activities

Use function machines that multiply by 10.

Enter a number → × 2 → × 10 = ?

Enter the same number → × 10 → × 2 = ?

What do you notice?

Try some divisions.

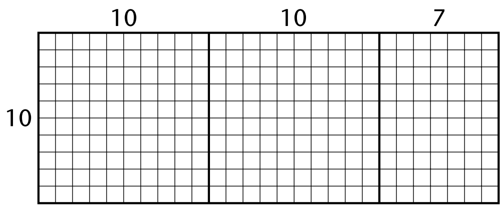
40 → ÷ 4 → ÷ 10 = ?

40 → ÷ 10 → ÷ 4 = ?

Try other starting numbers, such as 60, 20, 80, …

The function machine is a useful device for focusing attention on particular operations, in this case multiplication and then division. In the first part, pupils will notice that the order of multiplication does not matter – the effect of multiplying by 10 and then by 2 is the same as multiplying by 2 and then by 10. The machines can also work backwards, again illustrating that multiplication and division are inverse operations. But when dividing, unless pupils are able to cope with decimal fractions, you will need to be careful which numbers are entered.

Use a rectangular array to show multiplication, for example, of 27 × 10.



The area model is useful, especially to show how a number is partitioned into tens. It provides an image for pupils to visualise to aid mental calculation.

Use a multiplication chart.

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1000	2000	3000	4000	5000	6000	7000	8000	9000

Discuss the fact that the numbers on each row are found by multiplying the number above them by 10. So:

8 × 10 is 80, 40 × 10 is 400, and 500 × 10 is 5000.

If you skip a row, the numbers are multiplied by 100, so:

2 × 100 is 200, 70 × 100 is 7000.

Use the chart for dividing:

50 ÷ 10 = 5, 600 ÷ 10 = 60, and 4000 ÷ 100 = 40.

Extend the chart to show decimals by inserting decimals 0.1 to 0.9 above the numbers 1 to 9.

This chart is very helpful for showing multiplication by powers of 10. Going down a row has the effect of multiplying by 10, while going down two rows produces a multiplication by 100. Similarly, it demonstrates nicely that multiplication and division are inverse operations. Going up a row causes division by 10, and two rows division by 100.

Use multiplication diagrams. Ask pupils to find the missing numbers.

×	2		7
	40		
10		50	

Pupils are probably familiar with a conventional square of tables from 2 to 9. It can be more interesting to be given smaller arrays in a random order. Here pupils have to deduce that the first vertical column is part of the 2 times table; the number in the space below the 40 must be found by 2 × 10, so 20 must be entered. They can deduce that the middle column must be ×5 in order to get the 50.

Multiplying and dividing by single-digit numbers and multiplying by two-digit numbers

Once pupils are familiar with some multiplication facts, they can use these facts to work out others.

- One strategy that can be used is writing one of the numbers as the sum of two others about which more is known: $6 \times 7 = 6 \times (2 + 5) = 6 \times 2 + 6 \times 5$.
- Another strategy is making use of factors, so 7×6 is seen as $7 \times 3 \times 2$.
- A third strategy, for multiplication by 2, 4, 8, 16, 32, …, is to use a method of doubling, so that 9×8 is seen as $9 \times 2 \times 2 \times 2$.
- A fourth strategy to use is compensating, so that $2.9 \times 9 = (3 \times 9) - (0.1 \times 9)$, or $37 \times 19 = (37 \times 20) - 37 = 740 - 37 = 703$.

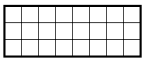
Since each of these strategies involves at least two steps, most pupils will find it helpful to make jottings of the intermediate steps in their calculations.

Expectations leading up to national curriculum level 5

12 × 4
93 ÷ 3
31 × 5
128 ÷ 4
46 × 3
135 ÷ 5
428 × 2
154 ÷ 7
3.1 × 7
48.6 ÷ 6
7.9 × 8
2.98 × 3
13 × 50
14 × 15
44 × 25
Change 15 minutes to seconds

Activities

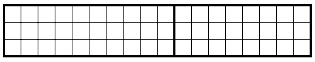
Use an area model for simple multiplication facts. For example, illustrate 8 × 3 as:



Ask pupils: 'How many rows are there? How many columns? How many squares?'

Encourage pupils to visualise other products in a similar way.

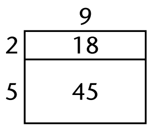
Extend this model to larger numbers, such as 18 × 3: split the 18 into 10 + 8 and use 10 × 3 + 8 × 3.



How many rows? columns? squares?

The rectangles give a good visual model for multiplication: the areas can be found by repeated addition (in the case of the first example, 8 + 8 + 8), but pupils should then commit 3 × 8 to memory and know that it is the same as 8 × 3.

Use multiplication facts that pupils know in order to work out others. For example, knowing 9 × 2 and 9 × 5, work out 9 × 7.



Area models like this discourage the use of repeated addition. The focus is on the separate multiplication facts. The diagram acts as a reminder of the known facts, which can be entered in the rectangles, and the way that they are added in order to find the answer.

Give pupils some examples which can be worked either by factors or by partitioning, such as:

27 × 12, 48 × 15

Discuss the special cases of multiplying by 25 and 50, which are easily done by multiplying by 100 and dividing by 4 or 2 respectively.

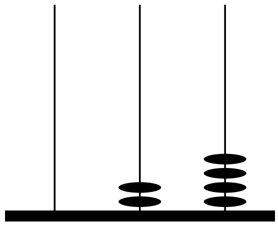
The use of factors often makes a multiplication easier to carry out.

Use base 10 material to model multiplication. Ask a pupil to put out rods to represent, say, 24.

Then ask for two more groups of 24 to be added, making three groups altogether. Make sure the pupil knows that as soon as there are 10 'ones' they are exchanged for a 'ten' or a 'long'.

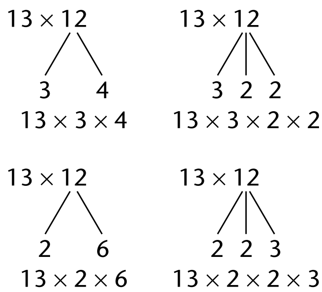
Record the result as 24 × 3.

Do the same, using a spike abacus. This time, as soon as there are 10 beads on a spike, they are removed and replaced by one bead on the spike to the left.



Base 10 material is an excellent model of the way in which we group numbers in tens. It lends itself to the concept of multiplication by 10 as each time a 'one' becomes a 'ten' and a 'ten' becomes a 'hundred'. Spike abacuses focus on place value, with multiplication by 10 resulting in the 10 beads on one spike being replaced by a single bead on the spike to the left.

Use factors to help with certain calculations. For example, do 13 × 12 by factorising 12 as 3 × 4 or 6 × 2:



Discuss which factors pupils prefer to use.

These diagrams can help pupils to keep track of the separate products when they split a number into its factors.

Doubling and halving

The ability to double numbers is useful for multiplication. Historically, multiplication was carried out by a process of doubling and adding. Most people find doubles the easiest multiplication facts to remember, and they can be used to simplify other calculations. Sometimes it can be helpful to halve one of the numbers in a product and double the other.

Expectations leading up to national curriculum level 5

Double any number to 20, eg double 18
Halve any even number to 40
Double any multiple of 5 up to 50, eg double 35
Halve any multiple of 10 up to 100, eg halve 90
Double any number from 1 to 50, then to 100
Halve any even number to 100, then to 200
$14 \times 5 = 14 \times 10 \div 2$
$12 \times 20 = 12 \times 2 \times 10$
$60 \times 4 = 60 \times 2 \times 2$
$36 \times 50 = 36 \times 100 \div 2$
$15 \times 6 = 30 \times 3$
One quarter of 72 = half of half of 72
$34 \times 4 = 34 \times 2 \times 2$
$26 \times 8 = 26 \times 2 \times 2 \times 2$
$36 \times 25 = 36 \times 100 \div 4 = 36 \div 4 \times 100$
$1.6 \div 2$

Activities

Ask pupils to halve a two-digit number such as 56. Discuss the ways in which they might work it out. Show pupils that, unless they can do it instantly, it will probably be better to partition it as 50 + 6 and to work out half of 50 and half of 6, then add these together.

Ask a pupil to suggest an even two-digit number and challenge other pupils to find a way of halving it. Some pupils might be able to respond to halving an odd number (47, say), by saying that it is 23 and a half.

Pupils should be familiar with halves of multiples of 10 up to 100, so that they can say instantly that half of 80 is 40 and so on. Larger numbers can be partitioned and the halving facts they know applied separately. Similarly, if they are familiar with the halves of multiples of 10 above 100, they can partition three-digit numbers in order to halve them. So they could halve 364, for example:
half of (300 + 60 + 4) = 150 + 30 + 2.

When finding 20% of an amount, say £5.40, discuss with pupils that it is easier to find 10% of it first and then double. 10% of £5.40 is £0.54, so 20% of £5.40 is £1.08.

Ask pupils how they would find 5% of £5.40. Then get them to work out 15% of £5.40.

This activity assumes that pupils are familiar with the idea of finding 10% of an amount, that is they know that 10% means 10 out of every hundred or one tenth. Then encourage them to use a range of methods for working out other percentages. For example, they might find 15% of £5.40 by finding 10% then halving that to find 5% and adding the two together. Or, having found 5%, they might multiply that result by 3. They could work out 17.5% by finding 10%, 5% and 2.5% and adding all three together.

Investigate doubling and halving number chains.

Ask someone in the class to choose a number. The rule that they are going to use is:

'If the number is even, halve it; if it is odd, add 1 and halve it.'

Go round the class and invite an answer. The rule is then applied to the answer and a new number is generated and so on, until they get to 1. Write all the numbers in the chain on the board or an overhead projector. For example:

14 → 7 → 4 → 2 → 1

Ask for a new starting number. Continue as before.

Number chains can be quite intriguing as it is usually not possible to guess what will happen. As more and more starting numbers are chosen, the chains can build up to a complex pattern. For example, the starting number 8 joins the chain above at 4; the starting number 13 joins the chain at 7. A starting number of 23, for example, goes to 12, then 6, then 3, then joins the chain at 2.

Fractions, decimals and percentages

Pupils need an understanding of how fractions, decimals and percentages relate to each other. For example, if they know that $\frac{1}{2}$, 0.5 and 50% are all ways of representing the same part of a whole, then the calculations

half of 40
$\frac{1}{2} \times 40$
$40 \times \frac{1}{2}$
40×0.5
0.5×40
50% of 40

can be seen as different versions of the same calculation. Sometimes it might be easier to work with fractions, sometimes with decimals and sometimes with percentages.

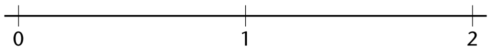
There are strong links between this section and the earlier section 'Multiplying and dividing by multiples of 10' (page 36).

Expectations leading up to national curriculum level 5

Find one tenth of 20, one fifth of 15, one third of 18
Find $\frac{1}{2}$ of 29, giving the answer as $14\frac{1}{2}$
Find $\frac{1}{2}$ of 150
Find $\frac{1}{2}$ of £21.60
Know that 0.7 is $\frac{7}{10}$ and that 6.7 is $6\frac{7}{10}$
Know that 0.5 is $\frac{5}{10}$ which is equivalent to $\frac{1}{2}$
Know that 0.25 is $\frac{25}{100}$ which is equivalent to $\frac{1}{4}$
Know that 0.75 is $\frac{75}{100}$ which is equivalent to $\frac{3}{4}$
Know that 0.03 is $\frac{3}{100}$
Find $\frac{1}{7}$ of 35
Find $\frac{1}{5}$ of 12.5 kg
Know that 10% = 0.1 = $\frac{1}{10}$; find 10% of £60
Know that 25% = 0.25 = $\frac{1}{4}$; find 25% of 16 km
Find 70% of 100 cm
Know that 0.007 is $\frac{7}{1000}$
Know that 45% = 0.45 = $\frac{45}{100}$; find 45% of £80
0.1×26
0.01×17
Find 17.5% of £5000
Find $\frac{1}{2}$ of $\frac{7}{10}$

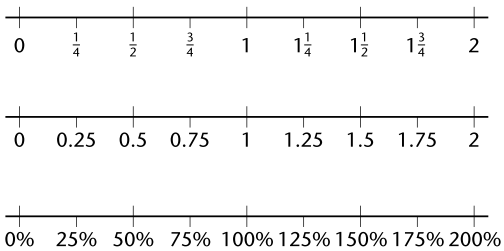
Activities

Draw a number line on the board, marking on it the points 0, 1 and 2:



Invite pupils to show where the fractions $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{2}$, $\frac{1}{4}$, $1\frac{3}{4}$ and $1\frac{1}{4}$ fit on the line. Ask what other fractions between 0 and 2 they could add to the line.

When they are familiar with fractions, draw a new line under the first one and ask for the decimals 0.5, 1.5, 0.25, 1.25, 1.75, 0.75 to be placed on this line. Repeat with a line for percentages from 0% to 200%. Then discuss the three lines.



Discuss the equivalence of, for example, $\frac{1}{4}$, 0.25 and 25%. Choose any number and ask pupils to call out the equivalents on the other two lines.

Number lines are useful for showing fractions as points on the number line between the whole numbers. This helps to move pupils from the area model for fractions, where they are seen as parts of shapes. The first part of the activity requires pupils to think about the relative sizes of fractions. Separate number lines with, for example, halves, quarters and eighths placed one under the other help to establish the idea of equivalent fractions. Gradually the lines can be built up to include more fractions, up to, say, twelfths. As in the example above, they can also be used to demonstrate the equivalence between fractions, decimals and percentages.

Write a sum of money on the board, for example £24.

Ask pupils, in turn, to tell you what $\frac{1}{2}$ of £24 is, then $\frac{1}{3}$, then $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$.

Then give fractions such as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{3}{8}$, $\frac{7}{12}$ and ask how pupils could calculate these fractions of £24.

In answering questions such as these, pupils will use the basic strategy of using what they already know to work out related facts. In this case, they will need to know how to find fractions such as $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of an amount and how to use this to find other fractions. For example, knowing that $\frac{1}{3}$ of £24 is £8 they can say $\frac{2}{3}$ is twice as much, or £16. Similarly, knowing that $\frac{1}{8}$ of £24 is £3, then $\frac{3}{8}$ is three times as much.

Put a percentage example on the board, say 25% of £60.

Discuss different ways of interpreting the question, such as $\frac{25}{100}$ of £60, or $\frac{1}{4}$ of £60.

Pupils might calculate $\frac{1}{4}$ of £60 by saying that $\frac{1}{2}$ of £60 is £30, and $\frac{1}{2}$ of £30 is £15. Alternatively, they might calculate 10% of £60, which is £6, so 5% of £60 is £3, and 20% of £60 is £12, so 25% of £60 is £12 + £3, or £15.

Ask pupils to find 17.5% of £60. Since they know that 5% of £60 is £3, they can work out that 2.5% of £60 is £1.50. Then they can find the total by adding 10% + 5% + 2.5%.

Invite pupils to suggest other examples.

Aim to give actual examples that pupils have seen in local shops or newspapers, to give a more realistic and motivating context.

It is important that pupils understand the basic fact that % means 'out of a hundred' or 'per hundred', rather than learn any rules about working with percentages. Examples of percentage calculations that can be done mentally can usually be worked out using this basic knowledge.

Further multiplication and division activities

Display a multiplication tables chart and see how many multiplication facts pupils know already. Shade these in on the chart.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50

… and so on to 100.

Make sure that pupils recognise that there is, for example, a row that starts 2, 4, 6, 8, and a column that has the same numbers. See how far they can continue the even numbers.

Point out the connection between the numbers in the ×4 column and the ×2 column. Ask them about the ×8 column.

Look at the ×5 column. Discuss the fact that these numbers end in a 5 or a 0, and that they are half the corresponding numbers in the ×10 column. Look at the patterns of the 9 times table.

Ask:

- 'Which numbers appear most often in the table. Why?'
- 'Which numbers less than 100 don't appear in the table. Why?'

Discussion of the patterns in a multiplication table can help pupils to remember multiplication facts. They might think that every number appears somewhere in the table, so it is helpful for them to think why numbers such as 11, 19 and so on do not appear (all the prime numbers after 7 are missing).

Get pupils to practise multiplying by multiples of 10 by putting a set of examples like these on the board.

5 × 10   12 × 10   8 × 20   13 × 20
18 × 20   35 × 30   120 × 20   125 × 30

Ask individual pupils to give the answer to any chosen question. Invite them to make up the most difficult example that they can do in their head.

Where appropriate, encourage pupils to use factors, so that 120 × 20 becomes 120 × 2 × 10.

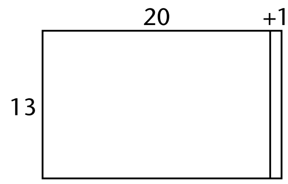
Put examples like these on the board. Ask pupils to suggest answers:

13 × 19   15 × 21   18 × 31

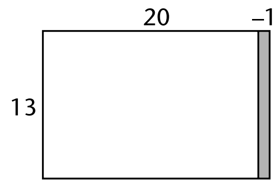
18 × 19   160 × 29

Discuss how they worked them out.

For 21 × 13, for example, they can think of 20 × 13 and add another 13. A rectangle like this might help them to visualise it:



and for 19 × 13:



The rectangles can help pupils to see how multiplication by 21, for example, can be carried out by multiplying by 20 + 1, and multiplication by 19 can be seen as multiplying by 20 – 1.

Write this multiplication on the board: 35 × 14.

Explain to the class that this might seem too difficult to carry out mentally, but that there are ways of making it easier to do. Challenge them to find a method that works for them.

They might choose to use the factors of 14, and say:

35 × 14 = 35 × 2 × 7

so they work out 35 × 2 to give 70 and then multiply 70 by 7.

Alternatively, they might partition the 35 into 30 + 5 and say:

35 × 14 = (30 × 14) + (5 × 14)

Discuss which of these strategies is more efficient.

6 Using a basic calculator

Calculators are powerful tools and, as with all tools, pupils need to learn how to use them properly. You may want to stress to the target pupils that it is not appropriate to use a calculator for calculations that can more quickly and reliably be carried out mentally. It is also important for them to learn how to use a calculator efficiently, including how to use calculator functions such as the square root key or memory.

To help pupils to judge when it is and when it is not sensible to use a calculator, it is useful to look from time to time at a mixed set of calculations presented on an overhead projector transparency and to discuss which of the calculations can be done mentally and which need a calculator.

You may need to control the use of calculators in mathematics lessons by distinguishing the times when pupils **must not** use a calculator, the times when they **must** use one, and the times when they **may choose** whether or not to use one. All three types of sessions have a place in the learning programme.

Calculator usage

The advantages of using calculators in mathematics lessons have been well rehearsed. For example, a calculator makes a useful teaching aid to help pupils to learn about numbers and the number system. A calculator designed for the overhead projector is particularly effective for generating discussion, in either a whole-class or group setting.

Calculators can also stimulate problem solving in mathematics and give opportunities for developing mathematical concepts that would otherwise be inaccessible to the pupils. A calculator allows them to work with real data, without the need for artificial simplifications. In this way, the calculator can open up a range of real problems that might arise in mathematics or in other subjects, such as science or geography.

Nevertheless, however a calculator is used, it does not do the strategic thinking for the user. It is important, before any step in a calculation is carried out, to decide what operation is appropriate. Equally important is the ability to interpret the display and to ask whether the result makes sense. Whenever pupils use a calculator, they should be encouraged to ask themselves questions such as:

- 'Roughly, what size of answer should I expect?'
- 'What numbers do I need to enter?'
- 'Do I have to add, subtract, multiply or divide these numbers?'
- 'In what order do I key in the numbers and operations?'
- 'How do I interpret the display?'
- 'Does the result make sense?'

The three important roles of the calculator are using it to:

- **work with real data and awkward numbers**
- **explore and gain understanding of mathematical ideas and patterns**
- **promote mental calculation skills.**

These uses are discussed in more detail in the following pages.

Calculator skills

Pupils need to be taught the technical skills of using a basic calculator. The target pupils, in particular, will not necessarily have these skills and some teaching time needs to be planned when the skills will be developed and practised. Pupils should know how to:

- clear the display before starting a calculation
- use the [+], [–], [×] and [÷] keys, the [=] key and decimal point to calculate with realistic data
- change an accidental wrong entry by using the [clear entry] key
- recognise a negative number output and use the [sign change] key where appropriate
- key in and interpret money calculations, for example:
  key in £4.35 + £3.85 as 4.35 [+] 3.85 [=], and interpret the result 8.2 as £8.20
  key in £6.30 + 85p as 6.3 [+] 0.85 [=], recognising that '0.' signals no pounds and only pence (alternatively, change money to pence and divide final answer by 100 to convert back to pounds)
- key in and interpret measurements of time, for example key in 4 hours 15 minutes as 4.25 hours (alternatively, change hours to minutes and work in minutes)
- find whole-number remainders after division with a calculator, for example be able to change 1000 minutes to hours and minutes
- interpret decimals, for example, know that a number such as 81.75 lies between 81 and 82
- interpret a rounding error, if it appears (eg interpret 6.9999999 as 7)
- key in fractions, recognise the equivalent decimal form, and use this to compare and order fractions
- read the display of, say, 0.3333333 as 'point three recurring', know that it represents one third, and that 0.6666666 represents two thirds
- use the square and square root keys
- select the correct key sequence to carry out calculations involving more than one step, for example $8 \times (37 + 58)$
- have a feel for the approximate size of an answer, and check it appropriately, for example: performing the inverse calculation or by clearing and repeating the calculation.

When they progress to using a scientific calculator, pupils will later need to be taught how to:

- use the memory and select the correct key sequence to carry out calculations involving more than one operation including brackets, for example $(23 + 41) \times (87 + 48)$
- use the fraction key
- use the relevant keys to find powers and roots
- use the π key.

Using a calculator for real data and awkward numbers

Enquiry into problems of genuine interest to pupils may involve calculations that are too awkward to deal with mentally. A mathematics programme in which all the numbers or measurements are simple enough to be dealt with mentally or with pencil and paper creates a limited view of mathematics and fails to give pupils an understanding of how mathematics is used in the real world.

Real contexts require a decision about which mathematical operations are necessary for the task. Examples in which pupils are directed, for example, to add or divide two given numbers offer practice, but not in the decision-making skills needed to solve real problems. When the development of these skills is the main objective of a lesson, it is appropriate to use a calculator. Pupils are then less likely to be distracted by any difficulties they may have in carrying out calculations.

A calculator will not interpret the display that arises from a calculation. If a pupil needs, say, to find how many coaches, each with a capacity of 54 people, will be needed to take 347 pupils on an outing, they may begin by using a calculator to divide 347 by 54. The display 6.4259259 is not, as it stands, a real solution to the problem and has to be interpreted. The pupil needs to know that the number of coaches has to be a whole number, in this case the next largest one: 7. If the pupil also wants to find the number of spare places on the coaches, further work has to be done. Multiplying 7 × 54, giving 378, the pupil can see that there are 31 spare seats.

The examples given below are related to the national curriculum at levels 3, 4 and 5, and are suitable for the target group of pupils as they enter and move through key stage 3. The examples include cases that require pupils to make decisions as to which operations they need to use, which calculator keys they need to press and in which order to press them. Many of them require pupils to interpret the resulting display on their calculator.

The least and greatest heights of girls in a class are 118.7 cm and 161.2 cm.
Find the range of heights of the girls in the class.

This is a realistic data-handling activity where the awkwardness of the numbers warrants the use of the calculator. The activity can be extended to other contexts in mathematics or other subjects where the range of values is of interest.

Collect some information on comparative prices for large and small amounts of any items. Ask, for example:
'Which is the better buy: 567 g of ketchup at 89p or 340 g at 49p?'

As the price of many goods is now given on labels as, for example, a cost per 100 g, comparisons are easier to make without calculation. But tasks such as these give opportunity to explore ideas of ratio.

A model village has houses that were built to a scale of 1 in 17.
One model shop is 47 cm high. What would be its real size?
The height of the village school is 11.85 m. What is the height of the model?

Pupils will need to decide whether they have to multiply or divide by 17 before they can use their calculator to carry out the appropriate operation. This kind of activity can be extended to map scales.

---

250 kilometres is approximately the same as 155.3 miles.
How many kilometres are there to 60 miles?

Pupils could think how they would use the information to find the number of kilometres in 1 mile (ie by dividing 250 by 155.3), and then multiplying by 60 to find the distance in kilometres.

To change a Celsius temperature to Fahrenheit, multiply by 1.8 and add 32.
To change a Fahrenheit temperature to Celsius, subtract 32 and divide by 1.8.

Using these rules, what is the Celsius equivalent of 83°F?
What is the Fahrenheit equivalent of 15°C?

The factor 1.8 arises from comparing the 180 degrees between the freezing point and boiling point of water on the Fahrenheit scale with the 100 degrees between the same two temperatures on the Celsius scale.

A machine makes 752 300 drawing pins every day.
These are packed in boxes of 125 pins.
How many boxes can be filled in one day?

While it may be possible for some pupils to carry out a calculation like this mentally, for others the numbers will be too large. In either case, the first task is to decide what sort of calculation to perform – in this case, 752 300 has to be divided by 125.

Petrol costs 78.9 pence a litre.
My car's petrol tank holds 41.5 litres.
How much does it cost to fill my car's petrol tank?

My car can travel about 340 miles on a full tank of petrol.
What is the cost per mile?

The first decision has to be to multiply 78.9 or 0.789 by 41.5 to get the total cost. Then the calculator display has to be interpreted. Knowing the cost for 340 miles, pupils will need to divide by 340 to find the cost per mile, and again interpret the display.

The planet Mars is 0.11 times heavier than Earth.
Jupiter is 317.8 times heavier than Earth.
How many times heavier is Jupiter than Mars?

To find how many times heavier Jupiter is than Mars, pupils will need to find how many times 317.8 is bigger than 0.11. Some pupils tend to assume all comparisons are made by subtraction, so they will need to think carefully about the importance of the word 'times' in the question. They need to find, in effect, how many times 0.11 will divide into 317.8. They will also need to interpret the display.

In 1955, it was decided that there were 31 556 926 seconds in that year.
How many minutes were there in the year? How many hours in the year?

Pupils need to take care in entering large numbers. After dividing by 60, they will need to interpret the calculator display to give an answer in minutes. At what stage should the interpretation of the display be made when calculating the number of hours in the year?

A computer printer can print 346 characters per second.
How many characters are there on this page?
How long will it take to print the page?

Pupils will need to decide how to arrive at the number of characters in a given piece of writing. Then the calculator will help them to arrive at an estimation of how long it will take to print.

Investigate the sequence: 1, 1, 2, 3, 5, 8, 13, 21, …
Each number is the sum of the two previous numbers.
Divide each number by the one before it: 1 ÷ 1, 2 ÷ 1, 3 ÷ 2, 5 ÷ 3, …
What do you notice if you keep going?

Some pupils may be familiar with the Fibonacci sequence; it has many interesting properties. In this case, as pupils carry out the successive divisions, their calculator display will eventually show numbers that always begin 1.618… . In fact, the ratio of adjacent terms in this sequence gets closer and closer to what is known as the Golden Ratio, which could be the beginning of further investigation.

The two examples that follow can be used as extension activities for more able pupils. They are typical of calculator activities that a teacher could devise for pupils working beyond level 5 in mathematics.

The distance from Brussels to Milan is approximately 785 miles.
An aeroplane flight took 1 hour 35 minutes.
What was the average speed of the aeroplane?

Knowing that the journey time is 95 minutes, pupils will need to divide 785 miles by 95 and then multiply the result by 60 to find the distance travelled per hour.

Six similar tubes of sweets were found to contain 46, 49, 47, 45, 51 and 49 sweets respectively.
What was the mean number of sweets in the tubes?
How many tubes of sweets can be filled from a container that has 5000 sweets?

The mean number of sweets per tube can be calculated in order to get an idea of how many would be filled from the large container. Alternatively, pupils could use the smallest number and the largest number of sweets to see what effect that has on the number of tubes that can be filled.

Using a calculator to explore and gain understanding of mathematical ideas and patterns

Calculators can help pupils to enhance their understanding of some fundamental mathematical ideas, such as negative numbers or place value. Calculators can also give opportunities to explore mathematical ideas that would otherwise be inaccessible to pupils, but which have intrinsic value. These include divisibility, recurring and terminating decimals and the effect of multiplying by numbers between 0 and 1. These ideas lend themselves readily to classroom discussion during key stage 3.

A calculator used with an overhead projector is invaluable in promoting whole-class discussion, as everyone can the see the display and the effect of entering or changing any number or operation. Tools for use with an interactive whiteboard also usually include display calculators.

Numbers in words

Each person in the group should have a calculator.
One person reads out the numbers.
The others enter them into their calculator, pressing the + key after each one.

Does everyone get the check number?

Set 1

One hundred and fifty-six
Two hundred and seven
Seven hundred and five
Three hundred and twelve
Six thousand, one hundred and forty
One thousand and eighty

Check number: 8600

Set 2

Twelve pounds sixty-three pence
One pound forty-two pence
Eighty-nine pence
Six pounds and six pence
Ten pounds and ten pence
Fifty pence

Check number: 31.6 (£31.60)

This activity allows you to check whether all the pupils in the group can translate numbers in words to numbers in figures. You may need to start with a shorter list for pupils at level 3. The activity can be extended to decimals, and to other measurements, including time.

Bull's eye

Choose a starting number (eg 17) and a target number (eg 100).
Find what number to multiply the starting number by to get as close as you can to the target number.

This gives pupils useful practice in estimation. A variation in which they may only multiply by whole numbers is a useful introductory activity before a lesson on long division.

Largest answer

Use the digits 1, 2, 3, 4 and 5 in any arrangement and one × sign.
You could have 241 × 35, 123 × 54 or 2431 × 5 etc.
What is the largest answer you can get?

This activity helps to focus attention on place value. Having to think about getting the largest product encourages pupils to think about where to put each number. Pupils can choose other sets of numbers, for example 5, 6, 7 and 8.

Prime factors

The number 74 865 can be written as the product of prime numbers.
What are they?

Pupils will need to use what they know about number in order to begin this activity. For example, they may decide to divide the number by 5 first.

Matching multiplications

Give pairs of pupils two sets of numbers, such as sets A and B.

Set A: 7, 15, 23, 34, 47, 53, 137
Set B: 782, 645, 371, 795, 1219, 959, 705, 345, 1802, 1081, 510, 1598

Players take turns.
The first player chooses a number in set B.
The player says which two numbers in set A will multiply together to make the selected number from set B.
The second player checks with a calculator.
If the first player is correct, the numbers in set B are marked with the first player's initials.
The winner is the first to mark four numbers in set B.

This game encourages pupils to consider what happens when two numbers are multiplied. For example, if one of the numbers ends in a 5, the product will also end in a 5. If one number ends in a 2, the product must be even. If one number ends in a 6, the product must end in a number that is a units digit in the 6 times table (6, 2, 8, 4 or 0). Such considerations help to improve mental calculation strategies.

Place value

Pupils work in pairs. One pupil enters a three-digit number on a calculator, for example 374. The second pupil suggests a change to one of the digits, for example 354, and challenges the first pupil to change the number to 354 with just one subtraction (in this case, – 20). The second pupil then challenges the first to make the new display into, say, 854, with just one addition (in this case, + 500). Each pair could set themselves a target number, such as 888 or 654, at which they will aim.

This game helps to reinforce the idea of place value. Pupils can announce, in advance, what operation they will use, in which case the calculator provides a check.

Matching divisions

Choose any number from this set: 1, 2, 3, 5, 8, 4, 6.
Choose another from this set: 3, 6, 8, 12, 10, 4, 6, 16, 32, 20, 15.
Divide the first number by the second.
Find some pairs that give the same answer.
Make up some more pairs that will give the same answer.

This activity helps pupils to work with decimal equivalents of fractions.

Multiplying by a number close to 1

Choose any two- or three-digit number.
Use the calculator to multiply the number by 1.1.
Try it with different starting numbers just greater than 1.
What can you say about the answers?
Now try multiplying by 0.9.
What happens now?

In this activity the calculator shows the effect of multiplying by numbers just greater than or less than 1. During discussion, pupils can explore the reason why this is so.

Exploring simple sequences

Use an overhead projector calculator to create sequences.

- Enter 10 – 1 = and keep pressing the = key.
  Count down with the display. Go past 0 and see what happens.
- Enter 103 – 10 = and keep pressing the = key.
- Enter 57 – 5 = or 12 + 5 = and keep pressing the = key.

Ask questions about each sequence.

- 'How does the units digit in the display change?'
- 'Why does this happen?'
- 'What will the next number be?'
- 'What number will appear in the display after ten presses of the = key?'
- 'Is it possible to exactly hit the number 20, 200, …?'
- 'After how many key presses will you pass 100?'

This activity works well with an overhead projector so that all pupils can see the display. Encourage prediction and reasoning by asking: 'What will come next?'

Using a calculator to promote mental skills

Opportunities to use a calculator to develop mental calculation skills often take the form of challenges between two pupils, one of whom works mentally while the other attempts to arrive at the solution more quickly with a calculator. This sort of activity encourages the use of mental methods; the challenge is to beat the calculator.

Beat the calculator

Pupils work in pairs with a pack of cards showing a set of additions of two whole numbers that are close together such as 35 + 36, 48 + 49, 125 + 126. The cards are placed face down in a pack. Players turn the cards over one by one. They then try to be the first to get the answer, one by mental calculation and the other with a calculator.

Alternative sets of cards can include division facts, or a set of simple multiplications such as 3 × 8, 21 × 7, 5 × 9, 13 × 3, …

This is a case where the calculator and mental calculation are compared for speed and accuracy. Games like these encourage pupils to increase their speed of calculation and the number of facts that they can recall rapidly.

Broken keys

Use only the 1, 0, 5, + and = keys.

Make each of these numbers. Press as few keys as possible.

16, 37, 88, 638

This well-known calculator activity encourages pupils to calculate mentally before they attempt to make the numbers using the calculator. Discuss different approaches.

Counting forwards and backwards

Set up two overhead projector calculators to act as simultaneous counters. For example, set calculator A to count forward in twos, starting at 0 (0 + 2 = = =), and calculator B to count backwards in threes from 100 (100 – 3 = = =).

Ask questions such as:

- 'What number will appear next on each calculator?'
- 'What happens to the sum of the pair of numbers? What happens to the difference?'
- 'Will the same numbers ever appear on both calculators at the same time? If so, which numbers?'

The role of the calculator in this activity is to give a context for asking a range of numerical questions that provoke mental calculation and stimulate class discussion.

The 100 complements game

Each player needs some counters of their own colour.

Players take turns. The first player chooses two numbers with a sum of 100 from this array.

50	65	26	30	19	85	10	96
35	39	36	18	31	52	73	46
55	28	37	77	22	12	17	20
51	81	59	1	69	70	9	25
72	90	78	88	48	5	82	39
11	75	64	49	50	63	38	99
80	27	45	61	23	74	41	54
91	4	15	62	95	83	15	89

The second player checks with a calculator. If the first player is correct, the player covers the two numbers with coloured counters.

The winner is the player who covers the most numbers.

This game provides motivation for mental calculation. The calculator acts as a referee; the pupil with a calculator is motivated to check the other player's accuracy.

Interpreting the display

Four books cost £6.
What does one book cost?

At a school concert, programmes cost 35p each.
How much money is collected when 140 programmes are sold?

A minibus has 12 seats.
How many minibuses are needed for an outing of 70 pupils and 7 teachers?

Simple questions like these can be used to develop pupils' ability to interpret the calculator display in the context of the original problem. For the first problem, the calculator display will show 1.5, which has to be interpreted as £1.50; in the second problem, the display of 490 has to be interpreted as £4.90. In the third problem, pupils have to round up the answer to the nearest integer.

Operations

Set up two calculators with two different functions. For example, one pupil has a calculator that has been programmed to perform + 2 (2 + =). A second pupil has a calculator programmed to multiply by 2 (2 × =). The class suggests a number for the first pupil to key into their calculator. The pupil shows the result to the second pupil who keys it into the other calculator and gives the class the final result. They then have to work out what the calculators have been programmed to do. The class can then predict the outcome for given inputs and the input for given outputs.

The calculator is used to give the motivation for the class to carry out mental calculation. Other functions can be tried.

7 Estimating and checking

Estimating and approximating

Mathematics is sometimes regarded as a subject in which all answers have to be either right or wrong. But there are many occasions in everyday life when an approximate answer is appropriate. For example, the newspaper headline: '75 000 fans watch Manchester United beat Arsenal' doesn't mean that exactly 75 000 people were there. One of two things may have happened. Either the exact number of people entering the grounds has been counted as they passed through the gate, and the exact number has been rounded to the nearest 1000 or 5000, or someone has estimated the number. Either is good enough for the newspaper headline.

Similarly, the value of π is usually taken as 3.14, because that is sufficiently accurate for the purpose, even though π is an irrational number that cannot be written in a precise form as a decimal.

It is often hard for pupils who have been accustomed to giving right answers to understand the idea that it is not always either possible or necessary to be exact. The idea of 'suitable for the purpose' needs time to develop.

The target pupils will probably be familiar with the idea of estimating in the context of measurement. The length of the classroom, for example, might be estimated in metres. The number of metres will be approximate but will probably be sufficiently accurate for the purpose. The pupils will probably be less familiar with the idea that every measurement is an approximation which depends on the sophistication of the measuring tool and the degree of accuracy required by the task.

Estimating the answer before embarking on a calculation helps to reduce later mistakes. So, before carrying out, say, 37×53, it is useful for pupils to recognise in advance that the answer will be more than 30×50 and less than 40×60. It may also be helpful to know that a closer approximation is 40×50. This strategy involves the idea of rounding numbers to the nearest ten or hundred, or whatever is appropriate. The fact that these approximations can be done mentally means that they give a way of getting a rough answer even when paper and pencil or a calculator are not available.

Some activities that help pupils to understand the ideas involved with estimation and approximation are given below.

Ask pupils to estimate the number of words on a page of a book or the number of dried beans in a transparent jar. They can see how good their estimate is by actually counting the objects, but help them to recognise that, just because their estimate is different from the actual number, it doesn't mean that their estimate is 'wrong'.

Suggest ways in which they might carry out the estimation. You may also want to discuss what would constitute a good estimate or a poor estimate, and how accurate their estimate might reasonably be (within 10, 20 or 100?).

Estimates like this can usually be made by considering areas or volumes. Pupils can estimate or even count the number of words in a typical line of text, and the numbers of lines on a page, and so arrive at their estimate, if necessary with the aid of a calculator.

Get pupils to estimate some lengths or distances in metres or centimetres. For example, estimate:

- the height of the classroom
- the width of a table
- the length of a corridor
- the distance from the school gate to the bus stop

Stress that, in making estimates like this, it is not possible to count. But it is useful to know a few facts, for example, that the height of an ordinary door is about 2 metres and that the width of a palm of a hand is about 10 centimetres. With these in mind, it is possible to visualise how many doors would fit the height of the classroom, and so on.

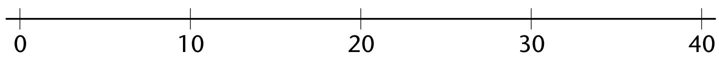
Get a group of pupils to play ‘four in a row’.

They will need two dice or spinners numbered 2, 3, 4, 5, 6, 7 and a 5 by 5 baseboard with 25 randomly placed multiples of 10 from 20 to 80. Pupils take turns to throw the dice and, using the digits in any order, make a two-digit number.

They should round their number to the nearest 10 and then place a counter on that number on the board. The aim of the game is to get four counters in a row.

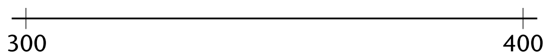
This game provides some simple practice in rounding to the nearest 10.

Use number lines such as:

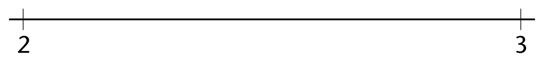


Ask pupils, in turn, where they would place the numbers 18, 25, 7, 32, and so on. Encourage them to explain how they decided where to place their number. What strategies do pupils use?

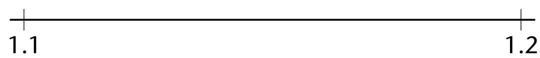
Extend the number lines used to include ones such as:



Where would you place 370, 320, ...?



Where would you place 2.5, 2.8, ...?



Where would you place 1.15, 1.13, ...?

Ask pupils to explain how they decided where to place each number. Discuss their strategies.

Get individual pupils or pairs of pupils to play ‘target’.

Give them some digits and ask them to decide how to place them in the boxes so the boxes give an answer as near to the target as they can. For example:

3, 4, 6	☐	☐	×	☐		Target 140
6, 7, 8, 9	☐	☐	×	☐	☐	Target 6000

This game involves an understanding of place value as well as an appreciation of the effect of multiplying by a one- or two-digit number.

Write some calculations on the board, such as 58 + 86 + 31.

Invite pupils to give what they think is an approximate answer. They might find two values between which the answer lies, by saying the answer lies between 50 + 80 + 30 and 60 + 90 + 40. Others might round each number to its nearest multiple of 10, giving, this time, 60 + 90 + 30.

Encourage pupils to make up some examples of their own and work on them in a similar way. You might like to get them to find which of the two methods is closest to the actual answer, and to say why.

Try other examples, such as 953 – 368, 63 × 87 or 953 ÷ 289. These examples pupils could clearly not do mentally, but they can get an approximate answer.

Play the ‘rounding game’ with a pair of pupils. They need a pile of single-digit cards, with no zeros. Pupils take turns to choose a card and place them to make three two-digit numbers. For example, they might get 73 + 56 + 18:

7	3	+	5	6	+	1	8

The first player rounds each two-digit number to the nearest ten and adds them together, getting:

70 + 60 + 20 = 150

The second player adds the original numbers (with a calculator if appropriate):

73 + 56 + 18 = 147

The player with the greater number scores the difference (3 in this case).

On the next round, players swap roles.

The winner is the player with the greatest score at the end of the game.

This game can be adapted. For example, use four or five cards to create multiplication of TU by TU; multiplication or division of HTU by TU, or division of ThHTU by TU. The game can be played by a whole class divided roughly into two teams. Each pupil gives their estimate or the result of the calculation by answering on mini-whiteboards.

Checking

Although in general it is preferable to get pupils to stop and think about the calculation before they embark on it, getting a rough idea of the size of the answer, there is also a strong case for encouraging them to think about the reasonableness of their answer when they have arrived at one. Experience suggests, though, that it is quite difficult to ensure that this actually happens.

There are three types of checking that are helpful: rough checking, partial checking and exact checking. Each type is discussed below and illustrated with examples.

Rough checking

This involves looking at the answer to decide whether it seems reasonable. A check can be made to see if the answer is the right order of magnitude, whether it is far too big or far too small. One might, for example, take a quick glance at a supermarket checkout bill, asking whether the total seems reasonable. Rough checks are usually done mentally and use the same procedures as for the approximations made before the calculation, as described earlier.

Mary bought some goods that cost £3.40, £6.87, £5.97 and £3.56.
The total was given was £81.63.
How do you know it is wrong?
What do you think the error was?

Rounding each item to the nearest £, the goods cost approximately
£3 + £7 + £6 + £4 = £20.
The error was to enter 68.7 rather than 6.87 for the second item.

The answer to 32 × 27 was given as 288.

How do you know it is wrong?
What is an approximate answer? What is 30 × 30?
What do you think the mistake was?

In this case, the error was to multiply 32 by 2 and 32 by 7, rather than multiplying 32 by 20 and 32 by 7.

How do you know the answer to 5.6 × 8.3 cannot be 4648?
What do you think the error was?

The error this time was to omit the decimal points and to multiply 56 by 83.

Partial checking

This is a way of deciding that certain features of the answer suggest that it is wrong. This time the question of whether the answer is reasonable makes use of some basic properties of numbers, such as whether the answer is even or odd, whether it is a multiple of 5 or 10. The final digit of the answer can be checked. In the case of multiplication, for example, a units digit of 6 can come from multiplying 6 and 1 or 2 and 3. Divisibility rules can also be useful: if a number has a factor of 9, for example, the sum of its digits is divisible by 9.

A pupil says that 7 × 8 is 55.
Why is the pupil wrong?

A pupil says that 13 × 13 is 167.
Why is that not possible?

A pupil says that 37 × 3 is 921.
Why is that not reasonable?

In the first example, since 8 is even, the product 7 × 8 must be even. In the second example, the units digit of the product must be 9, since 3 × 3 = 9. In the third example, the product must lie between 30 × 3 and 40 × 3 so cannot be 921.

Exact checking

This usually involves carrying out the calculation in a different way. This might involve turning a subtraction into addition or a division into multiplication. Or it might be possible to do the same calculation using a different method.

Check 13 + 27 + 35 by doing the addition in a different order.

Check 65 – 38 by adding: does adding 38 to the answer give 65?

Check 144 ÷ 16 by multiplying: does multiplying the answer by 16 give 144?

Check 12 × 15 by doing an equivalent calculation such as 6 × 30.

It is easy to use inverse operations on a calculator. For example, having multiplied 14.87 by 9 to get 133.83, immediately divide by 9 to see if you get back to 14.87.

Checking calculator work

Mistakes can easily be made when keying numbers into the calculator while carrying out a calculation. This is analogous to the errors that can be made between looking up a number in a telephone directory and then dialling that number. So checking an answer when working with a calculator is every bit as important as checking an answer when working with paper and pencil.

Some of the most common errors that calculator users make are:

- omitting a digit
- entering an incorrect digit
- entering digits in the wrong order
- entering a number from a list twice
- entering the wrong operation
- putting the decimal point in the wrong place.

One way of encouraging pupils to check their calculator answers is to get them to discuss the error that has been made in examples such as:

14.7 × 2.3 = 338
1001 × 13 = 77
56.71 – 33.47 = 33.24

Reinforcement

Encouraging pupils to estimate, to make approximations and to check their answers is a long-term process and one that has to be regularly addressed. Many teachers would agree that it is not easy to get pupils to see the importance of these aspects of mathematics. It is probably best to discuss them frequently and to make the most of any opportunity that arises.

Glossary

Approximation

A number that is close enough to an answer for the context or purpose.

Associative law

The description of operations in which the result of a combination of three or more elements does not depend on how the elements are grouped.

For example,
3 + 4 + 7 can be grouped as (3 + 4) + 7 or as 3 + (4 + 7) and
3 × 4 × 7 can be grouped as (3 × 4) × 7 or 3 × (4 × 7).

Division is not associative, as (12 ÷ 4) ÷ 3 ≠ 12 ÷ (4 ÷ 3).

Compensating

The process of restoring equality by addition and subtraction in order to make a calculation easier.

For example, 17 – 9 = 17 – 10 + 1.

Commutative law

The description of operations in which the order of the elements does not affect the result.

For example, 3 + 4 = 4 + 3 and 3 × 4 = 4 × 3.

Neither subtraction nor division is commutative:
4 – 3 ≠ 3 – 4 and 12 ÷ 3 ≠ 3 ÷ 12.

Complement (n)

A number that must be added to a given number to produce a specified total, usually 10 or a multiple of 10.

For example, the complement of 7 in 10 is 3, the complement of 96 in 100 is 4.

Consecutive integers

A series of whole numbers with equal steps of 1 between them.

For example, 3, 4, 5 or 11, 12, 13, 14.

Constant key

A key on the calculator (usually the = key) such that repeated pressing of it, after entering + or × and a number, adds or multiplies by that number repeatedly.

For example,
4 + 3 = = = … gives the sequence 7, 10, 13, 16, ...;
4 × 3 = = = ... may give the sequence 12, 48, 192, … or
12, 36, 106, … depending on the calculator.

Decimal fraction

A fraction in which the denominator is 10 or a power of 10, and in which the digits to the right of the decimal point show the number of tenths, hundredths, etc.

Digit

A single numeral from the set 0 to 9, which when used to represent a number in the place value system, is called a digit.

For example, the number 385 contains the digits 3, 8 and 5.

Distributive law	The principle that defines the way that two different operations can be combined. The distributive law of multiplication over addition or subtraction is illustrated thus: $(20 + 6) \times 7 = (20 \times 7) + (6 \times 7)$ $5 \times (50 - 2) = (5 \times 50) - (5 \times 2)$ The distributive law of division over addition and subtraction is illustrated thus: $(90 + 6) \div 6 = (90 \div 6) + (6 \div 6)$ $(75 - 9) \div 5 = (75 \div 5) - (9 \div 5)$
Estimate (v)	Use previous experience to make a judgement about the size of a specific number, quantity, measurement or calculation.
Equivalent	Numbers or measures that have the same numerical value.
Factors	Numbers that divide exactly into a given number. For example, 1, 2, 3, 4, 6 and 12 are factors of 12.
Fibonacci	An Italian mathematician (c 1170–1230), whose name is given to the sequence 1, 1, 2, 3, 5, 8, 13, 21, … in which each term, after the first two, is the sum of the two previous terms.
Fraction	The ratio of two whole numbers; also the result of dividing one integer by another integer. For example, $\frac{3}{4}$.
Function machine	An imaginary device that transforms one number into another by a defined rule.
Inverse operations	Operations that, when combined, leave the number on which they operate unchanged. For example, multiplication and division are inverse operations: $7 \times 3 \div 3 = 7$. Addition and subtraction are inverse operations: $7 + 3 - 3 = 7$.
Mean	A form of average in which the sum of all the elements is divided by the number of elements. For example, the mean of the numbers 3, 7, 8, 5 is found by adding 3 + 7 + 8 + 5 and dividing the result by 4.
Multiple	A number that has factors other than 1. For example, 12 is a multiple of 2, 3, 4 and 6.
Partitioning	The process of splitting a number into parts so that the sum of the parts equals the number. For example, 359 = 300 + 50 + 9.

Percentage

The number of parts per 100; the number of hundredths.

For example, 30% of something is 30 parts per 100, so that 30% is equivalent to $^{30}/_{100}$.

Prime number

A number that has exactly two factors, itself and 1. The number 2 is a prime number, but 1 is not.

Product

The result of multiplying one number by another.

Quotient

The result of dividing one number by another.

Rectangular array

An arrangement of numbers in the form of a rectangular block.

Remainder

When 14 is divided by 4, the quotient is 3 and the remainder is 2, because $12 = 4 \times 3 + 2$.

Rounding

Rounding is a way of writing a number with fewer non-zero digits.

For example, £16.25 rounded to the nearest pound is £16.

When the first digit to be ignored is 0, 1, 2, 3, 4 we round down; when the first digit to be ignored is 5, 6, 7, 8, 9 we round up. For example,
27 381 rounded to the nearest 100 is 27 400;
6.2489 rounded to the nearest tenth is 6.2.

Square root

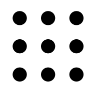
A positive or negative number that when multiplied by itself gives the original number.

For example, the positive square root of 25 is 5, since 5×5 or $5^2 = 25$.

Square number

A number that results from multiplying one number by itself; a number that can be represented by a square.

For example, 9 is a square number because $3 \times 3 = 9$, and 9 can be represented as:



Test of divisibility

A check to see whether one number divides exactly into another number.

Curriculum and standards

Audience	Teachers of mathematics at key stage 3, coordinators of mathematics across the curriculum, key stage 3 mathematics consultants, strategy managers and line managers, ITT departments
Circulation list	All schools with key stage 3 provision, key stage 3 mathematics consultants, line managers and strategy managers
Type	Curriculum guidance
Description	This booklet provides guidance on teaching mental calculation strategies to pupils who achieved level 3 or a weak level 4 in key stage 2 tests, with the aim of helping pupils to achieve level 5 by the end of key stage 3

Qualifications and Curriculum Authority
83 Piccadilly, London W11 8QA

Telephone: 020 7509 5555
Minicom: 020 7509 6546
Email: info@qca.org.uk
www.qca.org.uk

For more copies, contact:
QCA Orderline, PO Box 29, Norwich NR3 1GN
(tel: 08700 606015; fax: 08700 606017; email: orderline@qca.org.uk)

Price and order ref: £5 QCA/04/1358
Ref: DfES 0744-2004 G

ISBN: 1 85838 588 1

department for
education and skills

Raising Standards

INVESTOR IN PEOPLE

OUP 10-2004